"Application of Statistical Fractional Methods for the Analysis of Time Series of Currency Exchange Rates"

Irina Lvova

A thesis submitted in fulfilment of the requirements of

De Montfort University for the degree of Doctor of Philosophy

Faculty of Computer Science and Engineering

De Montfort University, Leicester

April 2006

PAGE MISSING IN ORIGINAL

ABSTRACT

This thesis examines some of the main issues of the application of statistical fractional methods for time series analysis of currency exchange rates. Key objects and structures of the foreign exchange market are described, and some of the fundamental approaches to financial theories are introduced. This thesis presents a description of conceptual approaches for the analysis of price characteristics of the foreign exchange market, provides theoretical grounds for simulation of financial processes, and introduces various models for currency exchange rates dynamics. The main features of fractional mathematical models are considered, and the use of the spectral parameter, included within a non-stationary fractional differential equation of time dependent order is argued as a key tool for the analysis of currency exchange rate time series.

This study presents a discussion of the fundamental problems and aims for analysing and handling of financial time series. The Ordinary Least Squares estimation is introduced, including its weaknesses and limits, and Rescaled Analysis is examined in full detail. The method of orthogonal regression is then proposed as a main research tool with justification for its limitations. The model for spectral parameter evolution, included within a non-stationary fractional differential equation of time dependent order is presented.

The analytical solution for the distribution function, characterising changes in time of the probability density function for the spectral parameter is obtained. The process of the evolution of the spectral parameter is statistically simulated, and some of the parameters, determining the shape of the distribution, are obtained. This thesis presents a practical method for finding the values of the spectral parameter from experimental data, based on the use of Fourier apparatus, using orthogonal regression, and elements of non-linear dynamics. Also, this research provides a comparison of the results for the theoretical and experimental estimation, and discusses the application limits. Parameters, accounting for the fluctuating components of the process of evolution of the spectral parameter, are obtained, and various criteria for identification and classification are introduced.

............To my family

ACKNOWLEDGEMENTS

Long before this work was finished, I was thinking about the acknowledgements. How to write this part of my work, how to express my thanks to those who were guiding me, who have grown me, helped with my education, whom I met? This is the most pleasant task, but also the most difficult for me.

This dissertation is the biggest achievement I have ever made in my research career, the result of years of studies and, I hope, a giant step forward in life.

First of all, I am indebted to Professor Jonathan Blackledge for the proposed subject for this study, background, supervision of my research on the first stages, when Jonathan was my first supervisor, and for his academic support during the next years.

I also want to thank my supervisors – Professor Gwynne Evans and Doctor Martin Turner – for their outstanding support, patience, criticism, ideas and guidance through the preparation of this thesis.

I am equally thankful to all members of my family – to parents and grandparents, who have made their best to raise and educate me, to my loving husband, whom I met in Leicester six years ago, and who was helping and supporting me these years.

Last but not least, I wish to thank De Montfort University for providing financial support and other facilities for my studies.

Finally I want to thank Charlotte Kitson, Joanne Cooke, Beverley Lambie and Marie Barker for coordination and administrative support during my study.

TABLE OF CONTENTS

List of Figures

List of Tables

Abbreviations

AGARCH	Asymmetric GARCH
APT	Arbitrage Pricing Theory
AR	Autoregressive (model)
ARIMA	Autoregressive Integrated Moving Average (model)
ARMA	Autoregressive Moving Average (model)
ARCH	Autoregressive Conditional Heteroskedastic (model)
CAPM	Capital Asset Pricing Model
CB	Central Bank
CHF	Swiss Franc
DEM	Deutsche Mark
DFT	Discrete Fourier Transform
DJ	Dow Jones Industrial Average
ECM	Efficient Capital Market
EGARCH	Exponential GARCH
EU	European Union
FFT	Fast Fourier Transform
FIGARCH	Fractionally Integrated GARCH
Forex	Foreign Exchange
FPK	Fokker-Planck-Kolmogorov (equation)
FBM	Fractional Brownian Motion
FNN	False Nearest Neighbour
FRF	French Franc
FTIID	Financial Times Industrial Index
FX-market	Foreign Exchange Market (Forex)
GARCH	Generalised Autoregressive Conditional Heteroskedastic (model)
GBP	Great Britain Pound
GMT	Greenwich Mean Time
HARCH	Heterogeneous ARCH
IID	Independent Identically Distributed
IMF	International Monetary Fund
ISO	International Organisation for Standardisation
IT	Information Technology

JPY	Japanese Yen
MA	Moving Average (model)
MIM	Minimal Inertial Manifold
NARCH	Nonlinear ARCH
NLG	Netherlands Guilder
OLS	Ordinary Least Squares (method)
PDF	Probability Density Function
PSD	Power Spectral Density
R/S Analysis	Rescaled Range Analysis
STARCH	Smooth-Transition ARCH
SV Model	Stochastic Volatility Model
TAR	Threshold AR
TGARCH	Threshold GARCH
US	United States
USD	United States Dollar

List of Main Notations

$A(q)$	Average speed of q
A_i	Multidimensional space
$\vec{a}$	Multidimensional vector
$B=(B_t)$	Brownian motion
$B(q)$	Speed of change of standard deviation in q evolution
b	Impact of incoming information
C_2	Correlation integral
D	Dispersion
D_2	Correlation dimension
d_T	Topological dimension
d_H	Hausdorff dimension
d_M	Minkovsky dimension
d_N	Geometrical dimension
d_C	Self-similarity dimension
E	Average of distribution
$f(q,t)$	Non-stationary distributions of q
$f_s(q)$	Stationary distributions of q
$f_e(q)$	Equilibrium distributions of q
$\tilde{f}$	Fluctuating component of $f(q,t)$
H	Hurst parameter
H_n	Price logarithms
h_n	Logarithmic gains
$\hat{K}_N$	Empirical kurtosis
k	"Physical costs" of the market participants, aimed at changing q
kq_{ec}^2	Intensity of the stabilisation activities of the CBs
M^k	Abstract mathematical object
$\hat{m}_k$	k^{th} empirical momentum
$\hat{m}_2$	Empirical variance
m	Reconstruction window
m_s	Enclosure dimension
$N(0,1)$	Normal distribution
R_n	Deviation of cumulative variables from its mean value within the past n years

$S = (S_t)$	Prices of currency exchange rates
S_t^a	Ask price
S_t^b	Bid price
$\hat{S}_N$	Empirical skewness
S_n	Empirical standard deviation
$\hat{s}$	Empirical standard deviation
$\hat{s} / \overline{S}_N$	Coefficients of variation
q	Spectral parameter (the amplitude spectral dimension)
q_c	Critical value of q
q_e	Equilibrium value, suitable for all market participants
q_{ec}^2	Activities of the CBs in relation to the activities of market participants
q_{ie}	Arbitrage attitudes of market participants
$q_{R/S}^N$	Spectral parameter (*R/S Analysis* for complete time series)
$q_{R/S}^m$	Spectral parameter (*R/S Analysis* for the reconstruction window *m)*
q_ρ^m	Spectral parameter (OLS estimation for the reconstruction window *m)*
q_m^m	Spectral parameter (mean OLS estimation for the reconstruction window *m)*
q_β^m	Spectral parameter (orthogonal regression (power spectrum) for the reconstruction window *m)*
q_q^m	Spectral parameter (orthogonal regression (amplitude spectrum) for the reconstruction window *m)*
$q_{\Delta X}^m$	Spectral parameter (values of the increments (properties of FBM))
q_σ^m	Spectral parameter (variance of increments (properties of FBM))
$q_q^\Sigma(t)$	Averaged time series of time series $q_q^m(t)$
$w(q)$	Average speed of the systematic change of q
$\tilde{w}(t)$	Fluctuating component of the speed $w(q)$
$\sigma = (\sigma_n)$	Standard deviation (volatility)
β	Spectral index (power spectral dimension)
ρ	Coefficient of cross-correlation (pairwise correlation)
Θ	Number (groups) of market participants with own investment horizons
τ	Delay parameter

CHAPTER I. Introduction and Background

1.1 AIMS AND OBJECTIVES

The main objective of this research is the analysis of time series for currency exchange rates through the examination of the behaviour of the spectral parameter, included within a non-stationary fractional differential equation of time dependent order. To do this, a new stochastic model of the spectral parameter evolution was developed, together with an extensive evaluation of methods for finding and estimating the spectral parameter. This focus on the problem of identification of time series of different currency exchange rates, and, in particular, on a more detailed analysis of multifractional parameters that can be used within time series, including spectral dimensions, the Hurst parameter and the correlation dimension. Using theoretical and practical research methods the key aims achieved within this thesis are:

1. To find the corresponding kinetic equation, and on its basis to develop a mathematical model for the evolution of the spectral parameter within currency quotations;
2. To evaluate the characteristics of a practical application for obtained equation of evolution of spectral parameter;
3. To perform statistical analysis of the evolution process of the spectral parameter, including the analysis of dynamic systems with asymptotic methods, and defining the determining parameters for the process;
4. To collect, systematise and estimate the experimental time series data for different currency exchange rates;
5. To develop a practical methodology and software to extract the spectral parameter for time series of currency exchange rate identification;
6. To estimate the spectral parameter for currency exchange rate quotations, and to compare these results with fractional dimensions, obtained using alternative methods;
7. To find the correlation dimension of the observed parameters, and consider the problem of distinguishing the chaotic and stochastic components of the process;
8. To compare the theoretical and experimental results, and determine the usability limits of the proposed model;
9. To determine the key list of parameters for the process of evolution of the spectral parameter;
10. To develop and experimentally test the methods for solving the time series identification problem.

1.2 THE STRUCTURE OF THE THESIS

This thesis consists of eight main chapters. Chapter II presents an analysis of the key features of contemporary financial relations, introduces the main objectives of this thesis and describes the activities of modern financial theory. This includes a description of the key factors, objects and structures that determine today's financial problems. On the basis of this the foreign exchange sector of the financial market is chosen as the main research data set for this study. Following initial analysis of the specific characteristics of the foreign exchange sector, the main applied research objects for this dissertation are introduced. In conclusion a description of conceptual approaches for the analysis of price characteristics of the foreign exchange market (the FX-market) is provided. In particular, direct analogies between physical and financial processes are made, and certain well-developed physical methods are applied to the financial processes.

In Chapter III theoretical grounds for the simulation of financial processes are considered. The analysis of the main physical approaches used for analysing the behaviour of the observed parameters is provided. A close correlation between the specified approaches is noted, and the importance of not only irreversible processes, but also of principles of self-similarity, invariance and symmetry is emphasised for self-organisation within macrosystems. After the notions of *fractals* and *fractional sets* are introduced, it is argued that the class of random fractals and multifractals is of key importance for this research. On the basis of results using statistical physics in combination with non-linear dynamics these methods are justified. It is also noted that physical and financial objects have fundamental differences as well as similarities, which impact on the approaches used for solving problems of financial analysis and the forecasting of price characteristics. Also, the classical theory of evolution of price characteristics is examined together with the key directions it has historically taken and the basic notions and terms of the Efficient Capital Market (ECM) theory are introduced as well as its weaknesses. This chapter provides structural classification for the mathematical models of economic and financial processes, and in particular the main types of mathematical models for price characteristics' dynamics. The specificity of linear models application, principal approaches to linear filtration of the data, and Fourier transform are also presented in this chapter. After the main constraints for application of linear models are stated, non-linear models are introduced with the aim of overcoming these limitations, and also explaining many of the phenomena and events, inexplicable in terms of linear models. This led on to mathematical models with properties of self-similarity (self-affinity). A number of comments is addressed regarding the issues of "independence and stability" in observed parameters (or their "dependence and normality"), applied in financial mathematics, and particularly for the analysis of the volatility fractional structure. Rescaled Range Analysis (*R/S Analysis*), which has been chosen as a key research tool for the time series, is described as well as a definition of Brownian motion. Together with Ito processes, and economical (geometrical) Brownian motion, the fundamental role of differential Fokker-Planck-Kolmogorov (FPK) equations, linking the theory of random processes and mathematical analysis, is noted. In conclusion, the main features of fractional mathematical models are considered, and the use of a non-stationary fractional differential equation of time dependent order $q(t)$ is argued for analysis, forecasting and simulation. The key role of the spectral parameter $q(t)$ in fractional equations is demonstrated, whose time series can be considered as an independent macroeconomic parameter for theoretical and experimental estimation in this research, is noted.

Chapter IV presents a discussion on the fundamental problems and aims for analysing and finding a solution to the identification problem. Different statistical methods of analysing and handling of time series are considered. The Ordinary Least Squares (OLS) estimation is introduced, including its weaknesses and limits. The method of orthogonal regression and *R/S Analysis* are examined in full detail, with justification for their application. Also, considered are other research areas including those, examining dimension estimation of the considered sets, and addressing the idea of the reconstruction of the attractor from experimental data, etc. Special emphasis here is made on the methods of dimension estimation of the dynamic system and its attractor. Methods of time series handling and analysis, emerged from systems with a fractional structure, including the use of the Fourier transform and spectral dimensions, are considered. This chapter concludes with a discussion on probabilistic dimensions and applied approaches to find solutions to distinguishing chaotic and stochastic sequences. This discussion is based on the use of the properties of the correlation integral.

Chapter V presents the model for the spectral parameter evolution, included within a non-stationary fractional differential equation of time dependent order. Using phenomenological probabilistic assumptions that are based on the stochastically determined nature of the processes of change of spectral parameter q for currency quotations, a kinetic equation of evolution of the parameter q is empirically obtained. The detailed analysis below shows that under certain assumptions this kinetic equation coincides with the FPK equation. Subsequently, using a rigorous statistical probabilistic argument, the boundaries of the practical application of the developed kinetic equation are examined. This chapter presents the analytical solution for the distribution function $f(q,t)$, characterising changes in time of the probability density function for q. The process of the change of the spectral parameter was statistically simulated (including the use of asymptotic methods), and some of the parameters, determining the shape of the distribution, were obtained.

Chapter VI provides preliminary results on data handling and the systematisation of experimental data for the considered time series of the financial market, supporting the use of fractional differential models, and the use of a non-stationary fractional differential equation of time dependent order in particular. The key methods for visual analysis are specified for these time series. Then a practical method is presented for finding the values of the spectral parameter from experimental data, based on the use of Fourier apparatus, using orthogonal regression, and non-linear dynamics. Using this method, this chapter presents experimental finding spectral parameters of currency exchange rates, including finding of the means of fluctuating components, and checking of the boundaries of this method. Results obtained with this method are compared with results obtained from the use of other methods. Some issues of distinguishing stochastic and chaotic components in the structure of the considered time series are examined in this chapter. Also, the size of correlation dimension of the considered time series is determined.

Chapter VII provides a comparison of the results for the theoretical and experimental estimation, and discusses the application limits. A specific practical application of the proposed method is presented including the extraction of appropriate parameters for the identification of those regions, which determine the behaviour of the dynamic system. In terms of the proposed model, the parameters, accounting for the fluctuating components of the process of evolution of the spectral parameter, can be obtained, and various criteria for identification and classification are introduced. This chapter also presents applied methods and approaches for solving time series identification. Relevant discussion on the use of these methods and approaches is then provided. The existence of a non-linear statistical relationship between various components of the process of the evolution of the spectral parameter is demonstrated, and the dominant impact of empirical volatility of the spectral parameter is experimentally supported.

Chapter VIII reviews the main results of this study, and highlights the key suggestions for future research.

1.3 ORIGINAL CONTRIBUTION

- This thesis provides the possibility of using statistical fractional methods for objective estimation of changes occurring in the time series of currency exchange rates.
- It is found that values of the spectral parameter, included within a non-stationary fractional differential equation of time dependent order, can be used as a criterion for the identification of anomalous changes in the time series of currency exchange rates.
- The method of finding the values of the spectral parameter from experimental data, based on the use of Fourier apparatus, using the method of orthogonal regression, and approaches of non-linear dynamics is proposed for this research. The possibility of practical applications of this methodology for identification of time series of the currency exchange rates is justified.
- It is found that errors, emerging from the use of a general form of the OLS estimation of the values of the spectral parameter for the time series for currency exchange rates, led to erroneous quantitative estimates, and results in dramatic misinterpretations in the nature of the occurring processes. This suggests OLS estimation is ineffective in this case. At the same time, the use of the method of orthogonal regression for amplitude spectrums is justified as an alternative.
- It is indicated that the proposed method for finding of the spectral parameter q_q can be considered appropriate not only for identification, but also for objective quantitative estimation of the mean values of the fractional dimension and the dimension of realisation of the time series.
- With regards to mean values, the obtained experimental results suggest the existence of a relationship between the values of the spectral parameter and the Hurst parameter in the form $q_q = q_{R/S} = H + ½$.
- It is experimentally obtained that sequences of values of the spectral parameter, as well as the sequences of the main quantitative characteristics of its evolution, maintain the most important properties of the initial quotations, including the properties of non-linearity, fractionality and statistical probability.
- The possibility of using statistical methods for the estimation of the process of the evolution of the spectral parameter is theoretically developed and empirically proved.
- On the basis of the FPK equation, the model for the evolution of the spectral parameter is theoretically examined, and the main parameters of the process are determined.

- In terms of the proposed model it is found that for the FX-market many locally stable equilibrium states exist. It is verified that for the FX-market there could be no "correct single model", and its description could be made through various integrated systems with different aims. There is no sense in looking for the "correct" state of the system, but the present state can be observed.
- It is found that, the values of the pairwise correlation coefficient of the direct and the inverse regressions in power spectrums of the process can be considered not only as the identifiers of the determining areas of the phase space of the system, but also as the parameter, characterising the predictability of the behaviour and the accuracy of forecasting in the described areas.
- The methodology for experimentation by finding the main quantitative parameters, determining systematic and fluctuating components of the process of the evolution of the spectral parameter, including complex indices for the whole process, is proposed. The existence of a statistical (non-linear and cyclic) relationship between systematic and fluctuating components of the process is experimentally supported. The possibility of interpretation of these parameters in physical, mathematical and financial/economical terms is provided.
- The dominating impact of empirical volatility of the process of the evolution of the spectral parameter, who's Probability Density Function (PDF) for the complete sequence is far from being normally distributed, is experimentally supported. It is found that while using complex indices for identification and classification, the change of empirical volatility is inversely proportional to $\frac{c}{\left(\hat{\sigma}_n^m(t)\right)^n}$, where $n \sim 4$.

CHAPTER II. Analysis of Key Features of Contemporary Financial Market

2.1 INTRODUCTION

Based on the analysis of the contemporary financial market and specifically its foreign exchange sector, this chapter justifies the choice of using time series of currency exchange rates between the main European and Japan currencies and the US dollar for the period between January 1971 and June 2000 as the main focus of this thesis. Those European currencies (except DEM) that were suspended after the introduction of the Euro, are excluded from this analysis, and due to the lack of sufficient statistical data on the Euro, this currency is also not considered within this thesis. The dynamics of currency exchange rates is analysed, and conceptual approaches to the analysis and forecasting of price characteristics on the FX-market are examined. For experimental estimation of the proposed research methods, the Deutsche Mark is considered as the most characteristic and dynamic currency over this period.

2.2 FINANCIAL THEORY: MAIN OBJECTIVES AND IMPLICATIONS

Nowadays, the term "finance" has become as uncertain and vague as the terms economics, mathematics, physics and programming. These disciplines have several independent spheres, each with their own set of problems, hypotheses, axioms, and research methods. Similarly, in finance it is possible to pick out a whole set of areas with fundamentally different final aims and ways of problem solving. According to modern points of view [1, 2, 3], financial theory investigates the characteristics of financial structures and focuses on the optimal allocation of financial resources when considering time, risk and environmental factors (usually random), using financial techniques and operations. This allows us to identify the following key implications of financial theory: financial data and structures analysis; financial forecasting; optimisation and managerial decision-making at all levels of financial objects, structures and systems.

Traditionally, financial theory and practice focuses its main attention on financial data and structure analysis [4, 5, 6], which is fundamental to simulation, forecasting and management. A variety of models and methods of financial data analysis has close relationship with other branches of science, particularly with applied mathematics and programming [4,..., 7]. Thus, the development of existing, and the creation of new, models and methods of analysis are among the main prior research directions in this field, which will provide future scholars with additional information for further research. In particular, this includes obtaining new complex parameters (characteristics, variables, measures, estimates, etc.) for financial analysis, finding a link between these new variables and previously well-known parameters. The field of financial forecasting [4,…, 12] varies from relatively simple methods for short-term financial forecasting to rigorous open-ended research questions [10, 11, 12], including, an understanding of the fundamental barriers to what lies beyond the available research methods that cannot be addressed by financial analysis in general [10]. The question of optimisation and managerial decision-making in terms of financial theory is separate from the questions described above. On the one hand, it is based on the results obtained from the first two research fields [13], but, on the other hand, it is also based on approaches from related fields of research [14]. These, in particular, include not only economics, corporate governance, programming, etc., but also administration and optimisation methods, risk analysis and decision making in an unstable environment with multiple alternatives [15].

The main focus of this thesis is on the analysis of financial data. Questions of forecasting, managerial decision-making, optimisation and risk analysis are not examined here. Although these research fields of financial theory are quite distinctive, they are linked to each other. Moreover, in terms of the methods applied, they are linked not only with economics [16, 17, 18], but also with natural sciences [7, 9,…, 12, 15, 19,…, 28]. This correlation results from both the peculiarities of modern financial relations and the unique approach to analysis and simulation of processes such as financial, economics, chemical and physical. Another aspect, linking different research fields of financial theory lies within those key objects and structures that explain the specificity of financial problems.

2.3 Financial Relations: Determining Factors, Objects and Structures

The key objects and structures in finance can be specified as individuals (physical persons), firms (juridical persons), intermediary structures (a mix of natural and juridical persons) and the financial market itself [6]. Taking into account [1, 2, 3], the characteristics of interaction of the described structures and financial relations can be thoroughly presented with Figure 2.1.

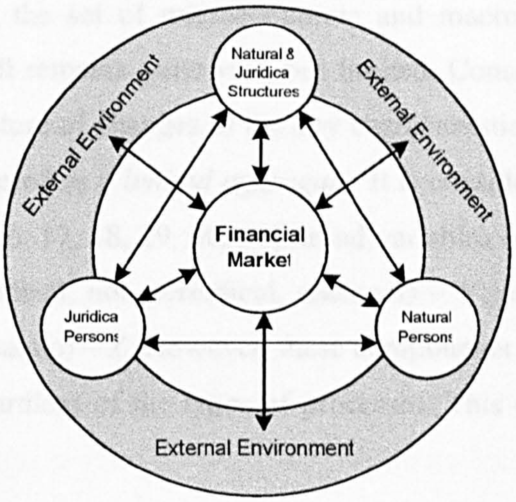


Figure 2.1. A Diagram of Financial Relations

Figure 2.1 shows that in financial relations there are a number of dynamic systems with different levels of complexity. These systems consist of numerous elements, and each of these elements has a very large number of degrees of freedom. Such systems are characterised not only by fundamental economic processes, such as capital accumulation, supply and demand, but also with other non-economic processes. Figure 2.1 allows both internal and external structures to be classified in financial relations. Among the major factors, determining the structure of these financial relations, it is worth highlighting: economic, legal (juridical), social (political), administrative and physiological factors.

The financial market at micro and macro levels, while exchanging information with other objects and structures, performs in *uncertain conditions* under the great bulk of a constantly changing external environment. This allows for the consideration of the financial market as a *large complex open dynamic system*. In the general case, it is not possible to study the financial market separately because all objects, structures and factors are interrelated: financial and non-financial, internal and external, market and non-market. As all these sectors influence each other, it is necessary to take into account different types of relation mechanisms. For instance, between-markets analysis has been aimed at studying the external factors rather than the internal financial information [29]. It is possible to proceed with the structural specification of relations (Figure 2.1), however, it is already clear that the financial market plays a key role in financial relations, justifying its choice as the main objective for this research.

Figure 2.1 shows another important property of the structure of financial relations – that is the limited nature of interaction between objects and structures. The range of financial relations can still be presented in the form of a dynamic system with its own restrictions, rules and behavioural features. In general, the set of microeconomic and macroeconomic factors, financial techniques and relations still remains quite vast, but limited. Consequently, a number of components, determining the nature of changes of the key characteristics, underlying financial relations, should also be considered as a *limited aggregate*. It is considered that this aggregate consists of [1,..., 6, 8, 13, 14, 16, 17, 18, 29, 30, 31]: trend variables (slowly changing, long-term) – X; cyclic variables (periodical, non-periodical, seasonal) – Y; and fluctuating variables (irregular, stochastic or/and chaotic) – Z. However, these components can give observed empirical data S in various ways regardless of the types of processes. This can be represented with the following formula:

$$S = X * Y * Z, \qquad (2.1)$$

where various mathematical operations could be implied as the operator "*".

Fundamentally, this gives an opportunity to describe and analyse the behaviour of key objects and structures of financial relations (at micro or macro level) with models and quantitative methods that take account of their various specific aspects. To justify the choice of particular applied process-specific financial mathematical models, and to be able to adequately interpret the results which will be obtained afterwards, first it is necessary to examine specific aspects of the objects considered (mostly of the financial market).

2.4 THE FINANCIAL MARKET AND CURRENCY EXCHANGE SECTOR

The financial market *is not a homogeneous structure*, and has certain specific sectors. In particular, in the financial market it is possible to distinguish four market sectors: currencies, goods (precious metals), bonds and shares [6, 30, 31]. This work focuses on the study of the currency exchange sector, as the dominant and most dynamic sector within the financial market [6, 46,...,49]. This choice is determined by the fact that, on the one hand, transactions on the foreign exchange market can be explained with the simplest microeconomic scheme – barter, where only two goods are exchanged. On the other hand, the foreign exchange market is characterised with the existence of a complex macroeconomic reaction of its sectors due to changes in external environment and other markets and the introduction of exchange regulation schemes [46].

For a long time in the international monetary and credit system among the main means of paying were precious metals, particularly gold [6, 46, 47]. The era of the "Gold standard" started in 1821 and totally disappeared in 1971, when the US Treasury stopped purchasing and selling gold at a fixed price. The development of international economics led to the establishment of currency blocks for different countries, which agreed on exchange rates and other aspects of monetary and credit policy. A good example is the Breton Woods Agreement, which was organised in 1944. According to this agreement, exchange rates were fixed in advance and after that could vary only within ± 1% from those predetermined values (based on US dollars). Also, to let the agreement operate, the International Monetary Fund (IMF) was established. However, after the 1973 financial crisis affected the world's major currencies (US dollar, Deutsche Mark and Japanese Yen), the era of the Breton Woods Agreement was exhausted, and flexible exchange rates replaced these fixed exchange rates. In March 1979 the European monetary system was introduced. This has embraced most of the European Union (EU) countries, and settled possible variations of exchange rates within ± 2.25% from the official predetermined values, decided by member countries. Central banks in countries where exchange rates were expected to change above the allowed ± 2.25% were obliged to prevent this, thus keeping the stability of all the members' exchange rates. The last fact explains why these systems are also called "adjusted system of floating exchange rates". Among other examples of currency blocks are agreements between different countries that fix their exchange rates to some other "hard currency". This thesis does not examine these blocks and the resulting exchange rates, although for more information on this issue it is possible to refer to [47].

From the practical application point of view, the "adjusted system of floating exchange rates" is the most interest period of time for this research. From the strategic point, this system is characterised with the presence of permanent control and elaboration of coordinated correcting activities of the CBs (central banks/control centres), aimed at the stabilisation of the exchange rates, rather than with the existence of strict limitations on the foreign exchange market. This considers the foreign exchange sector as a large and relatively complex open dynamic system, which is constantly controlled and managed by the CB issues on the basis of regulated traffic of different types. To find the main properties of such system, we will consider the key operation features of the foreign exchange market over a given period of time.

2.5 FOREIGN EXCHANGE MARKET AND ITS KEY FEATURES

Unlike many other exchanges the foreign exchange market (i.e. Forex, or the FX-market) operates cyclically and non-stop, and has its own interesting features. The FX-market is *international* and it is not localised and comprises of the whole network of banks and exchange offices around the world [6, 29, 30, 31, 46,..., 50]. Nowadays, there are a number of specialised agencies, like Reuters, Telerate, Bloomberg, etc. [49, 50], which provide numerous online information on a twenty-four hour basis. There are also special software packages [49, 50] for handling this information, with embedded automatic chart options [29, 30, 31, 48, 49, 50]. Also together with price characteristics, data on volumes of trade, open interest, and other (macroeconomic) characteristics, can be used for the analysis of quotations. This indicates a time-specific nature of the FX-market, and also means that there is a whole space-time structure that combines currency auctions' results in different countries at different moments of time. Results of foreign exchange operations are examined over a wide range of time intervals – from a few seconds up to centuries [6, 29, 30, 31, 50]. One of the most sizable databases on the FX-market activities belongs to Olsen & Associates [51,..., 57].

Analysing any graphs of "ticks" (one-minute, hourly, daily, weekly), it is possible to see that for a period of time the price does not change unless one of the traders offers a new price – at this moment a new "tick" happens. This allows us to conclude that price changes on foreign exchange are discrete and the next two questions emerge: statistically, what is the length of the time intervals between ticks and how do prices change? Table 2.1 provides an insight to the activities on the FX-market [6] for the period of 01.01.1987 – 31.12.1993.

Table 2.1. The FX-Market Activity

Currencies	Total number of ticks in the database	Average number of daily ticks
DEM/USD	8 238 532	4500
JPY/USD	4 230 041	2300
GBP/USD	3 469 421	1900
CHF/USD	3 452 559	1900
FRF/USD	2 098 679	1150
JPY/DEM	190 967	630
GBP/DEM	96 537	320
NLG/DEM	20 355	70

Abbreviations in Table 2.1 are the standard abbreviations of the International Organisation for Standardisation (ISO) for currencies (code 4217), denominated in US dollars (USD) or Deutsche marks (DEM). The average number of ticks per day is calculated on the basis of 52 weeks with 5 working days per year. These results show the high frequency of changes in currency exchange rates on the FX-market, especially with respect to the US dollar. At the same time, the Deutsche Mark has to be recognised as the most dynamic currency, which justifies its choice for this thesis. Also, important is the fact that only contemporary methods of continuous data collection and analysis allow for the determination of the key features in the behaviour of the exchange rates. Among them are non-linear tendencies in the dynamics of exchange rates;

the existence of after-effects (i.e. when exchange rates "remember" the past); high frequencies in the behaviour of exchange rates (changing chaotically in time), which disappear after doing time or phase discretisation (sampling) of variables [6, 52]. Figures 2.2 and 2.3 demonstrate intraday activities on the FX-market for the period of 05.10.1992 – 26.09.1993 [6, 52].

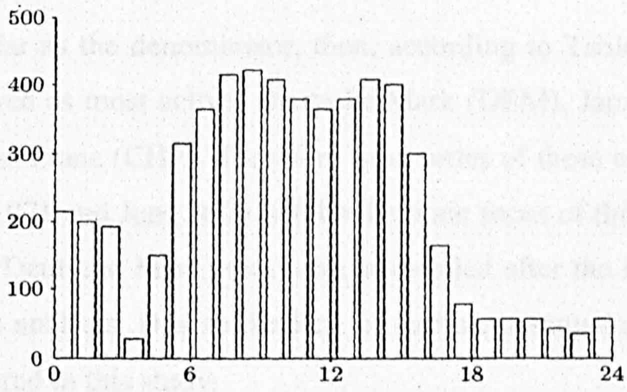


Figure 2.2. Average Intraday Tick Activity for DEM/USD

These results show that the FX-market operates non-stop 24 hours a day, and during the day the changes (ticks) are heterogeneous, which is the result of the Earth's rotation and time change around the globe. For example, the FX-market activities go down when it is lunch time in Tokyo and night in Europe and America. Following [6, 52], Figure 2.3 shows the FX-market daily activities (ticks) for DEM/USD exchange rates.

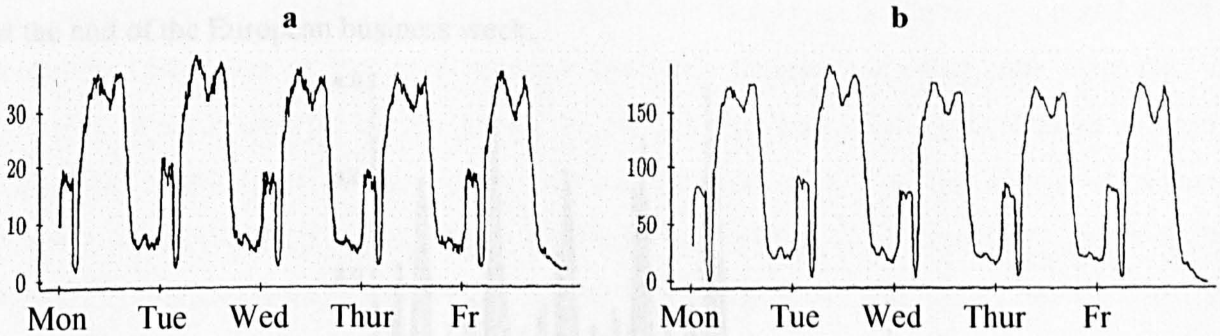


Figure 2.3. Average Daily Monday – Friday Tick Activity for DEM/USD

a: 5 Minute Intervals; **b:** 20 Minute Intervals

Figure 2.3 shows that there is heterogeneity in tick activity of a day-to-day cyclic variation, caused by the Earth's rotation. It is possible to distinguish three peaks of activity, typical of the three geographical areas. Traditionally [6, 48, 49, 54], the following three zones of stock activities are recognised: East-Asian zone (including Pacific zone) with the centre in Tokyo (9:00 p.m.-7:00 a.m.(GMT)); European zone with the centre in London (7:00 a.m.-1:00 p.m.); and the American zone with the centre in New York (1:00 p.m.-8:00 p.m.). As can be seen from Figure 2.3, peaks of activities in Europe and America are nearly the same, although the American zone of foreign exchange is known as more “aggressive”. That’s why we consider the European zone for this research. This notion is also supported by many scholars [49], who recognise the European market as the most “stable” and “predictable”. It is believed that this notion will be helpful and the European market will be more “suitable” for the subsequent analysis.

If we take the US dollar as the denominator, then, according to Table 2.1, the following four currencies can be viewed as most active: Deutsche Mark (DEM), Japanese Yen (JPY), British Pound (GBP) and Swiss Franc (CHF). Therefore time series of these exchange rates for the period between January 1971 and June 2000 will be the main focus of this thesis. Those European currencies (except the Deutsche Mark) that were suspended after the introduction of the Euro, are excluded from this analysis. Due to the lack of sufficient statistical data on the Euro, this currency is not considered in this study.

Another interesting feature of the FX-market – is its very active behaviour during the business week combined with inertia at weekends and holidays. Figure 2.4 is presented below to support this notion on the example of CHF/USD exchange rates [6, 54]. The graph describes the weekly market activity over 168 hours referring to one typical business week. It shows that there is an intraday heterogeneity of data together with daily cyclic recurrence and intraweek activity fluctuations. The beginning of a new week in the East-Asian zone stipulates the increasing activity at the end of the European business week.

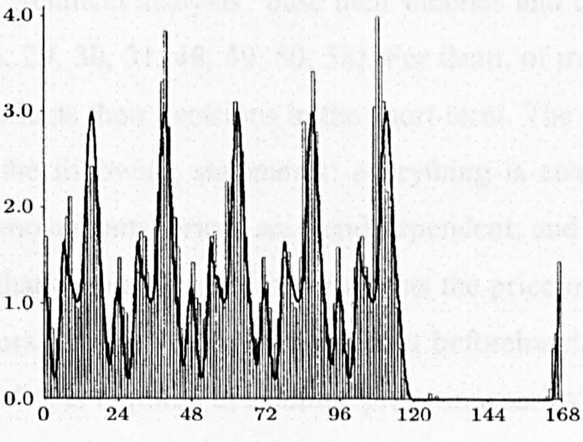


Figure 2.4. The Activity of CHF/USD (Weekly, GMT)

The above discussion on the FX-market characteristics shows the time dependent nature of currency exchange rates, and allows for the consideration of empirical evolution of data in both the astronomical and working days. Since the latter representation of data makes the exchange process more or less continuous, it is reasonable to use this within this research. However, the question of statistical changes of currency price characteristics, their time horizon, and resulting financial analysis and forecasting are still to be examined. (Investment horizon is a period of time required for a trader to analyse information and make a decision.) Understanding of the importance of these issues requires detailed consideration, starting from a review of the conceptual approaches to financial analysis and forecasting of price characteristics.

2.6 Conceptual Approaches to Analysis and Forecasting of the FX-Market Price Characteristics

Depending on the type of information available, it is possible to distinguish two approaches, used by participants, to financial analysis and price forecasting – *fundamental* and *technical* [6, 29, 30, 31, 46,..., 50, 58]. Both of these approaches try to solve one problem of forecasting future market price movement. Advocates of the fundamental approach look for cause, and "technical" researchers look for effect. "Fundamental" researchers build their theories upon the "global" state of the economy. The most important information for them is on future prospects, since rational behaviour of market participants is assumed by these "fundamental" researchers. Following [17, 46, 50] the traditional set of "fundamental" indices affecting currency exchange rates in a medium-term forecast consists of: gross domestic product; purchasing-power parity; interest rates; inflation and unemployment levels; trade balance of the country; etc. "Fundamental" researchers also consider a whole set of short-term factors, such as acts of God and political instability. Advocates of "technical analysis" base their theories and conclusions upon the "local" state of the market [6, 29, 30, 31, 48, 49, 50, 58]. For them, of most importance is "crowd behaviour" as this factor affects their decisions in the short-term. The main axioms of technical analysis are founded on the following statements: everything is concerned in market movements, in particular price movements; prices are trend dependent; and history repeats. The core idea of the first axiom is that each and every factor affects the price over time – economic, political, psychological factors, etc. – is taken into account beforehand. Consequently, from the point of "technical analysis" it is required to examine price time series for analysis and forecasting. According to the second axiom, the introduction of a *trend* term, that implies a certain direction for price movement, is essential. In modern econometrics and financial analysis, *trend* is a recognised parameter [4, 5, 6, 8, 13, 18, 29, 30, 31, 48, 49, 50, 58]. The following three trend types are distinguished in technical analysis: bullish – prices go up; bearish – prices go down; sideways (trading range) – prices are (almost) not changing over time. The third axiom of technical analysis looks natural and is evident from a human point of view. Due to this, technical analysts assume if some particular types of analysis and forecasting were working in the past, they will also be working in the future, because this is based on the stability of human perceptions. In other words, future forecasting starts from understanding the past. The postulates considered for technical analysis leads to the conclusion that market processes are regular dependent (usually cyclically and non-linearly) and thus it is possible to successfully analyse and predict the price. Technical analysts were the first to start (although at the empirical level) using methods of non-linear dynamics, including methods of fractal geometry [29, 30, 31, 49, 50, 58,..., 61]. On this basis a number of practical methodologies (including graphical analysis of charts) were developed and classified together with key determining criteria (indicators) [62]. Contemporary non-linear dynamics and fractal geometry have positively influenced not only practical methodologies directly, but also affected them conceptually.

According to Peters [58], during 1920 – 1940 “fundamentalists” and “technicians” became the two main groups of financial market analysts. During the 1950s, another group of Louis Bachelier followers, named “quants” (quantitative) analysts, added to the two groups. “Quants” analysts developed their theory and methods on prices’ random walk hypothesis and the theory of random processes, particularly on the theory of Brownian motion [4,..., 8, 13, 24, 28, 32, 37, 41, 44]. However, quantitative analysts are closer to “fundamentalists” than to “technicians”.

This grouping of analysts is not fixed since followers of one philosophy use the results and progress achieved by the other groups. It shows that the investment horizon is a key indicator by choosing one of the approaches for finding solutions to a particular practical problem. In cases of short-term time horizons (minutes, hours, days), “technical” information is crucial, but when the time horizon increases (days, months, years) the key role shifts to “fundamental” information. Apparently, methods of “quants analysts” are more appropriate for medium-term time intervals (hours, days, months). This notion is supported by the analysis of real financial data [8], where the same parameters behave as trend, cyclic or random walk parameters on different time horizons. This is also true when choosing the models describing the price characteristics behaviour on the FX-market [6].

A close correlation between the length of the investment horizon and a conceptual approach to analysis and forecasting of price characteristics at the FX-market confirms that the problem of finding the right time interval, used as the investment horizon parameter, and as the sampling parameter for the time series is non-trivial and requires in-depth analysis. The classification of financial processes, presented in Section 2.3, (almost) completely coincides with a similar classification of physical processes [7, 9,..., 12, 15, 19, ..., 28]. Therefore, it seems reasonable, along with purely economic and financial methods [1,..., 6, 8, 13, 14, 16, 17, 18, 29, 30, 31], to try to apply the well-developed physical methods directly to financial processes [7, 9,..., 12, 15, 19,..., 28]. Since these approaches can be successfully applied to solve problems by finding the rules of change of the observed parameters (not only physical parameters) in time and space, it is worth considering them in full detail in this work as well [7, 9, 10, 11, 23, 26].

2.7 CONCLUSION

The financial market is not homogeneous and takes a central position in the structure of contemporary financial relations. Interrelations and processes lying in the core of financial relations have trend, cyclic and fluctuating components in their structure. They organise bounded aggregation that allows for the possibility of quantitative analysis and description. The financial market is functioning in uncertain conditions, continuously exchanging financial information with other objects and structures under an external environment, which allows it to be considered as a complex dynamic open system. The currency exchange sector takes the dominating and the most dynamic position in the structure of the contemporary financial market. The key features of the contemporary FX-market allow us to consider this market as a large and relatively complex open dynamic system, which is constantly controlled and managed by the CBs on the basis of regulated traffic of different types. Characteristics of the contemporary FX-market allow us to focus on analysis of the European zone currency exchange rates. Time series of the currency exchange rates: Deutsche Mark (DEM), Japanese Yen (JPY), Great Britain Pound (GBP) and Swiss Franc (CHF) for the period between January 1971 and June 2000 will be the main focus of this thesis. Those European currencies (except the DEM) that were suspended after the introduction of the Euro, has been excluded from this analysis, and due to the lack of sufficient statistical data on the Euro, this currency will also not be considered in this study.

There is a close interrelation between the investment horizon parameter and a conceptual approach to the analysis and forecasting of price characteristics of the FX-market. For up-to-date empirical analysis on the FX-market it is impossible to investigate many observed non-trivial effects and phenomena, and also difficult to establish cause-and-effect relations, which have to be taken into account when doing objective quantitative analysis, particularly for justification of the choice of the determining time intervals. Thus, we have to consider general theoretical grounds for the simulation of financial processes, specifically related to the foreign exchange sector.

CHAPTER III. Theoretical Grounds for the Simulation of Financial Processes

3.1 INTRODUCTION

Based on the theoretical analysis of the simulation of financial processes, this chapter postulates that new effects and phenomena arise when re-considering recognised problems from a new perspective and using modern tools and models, which are based on previous studies and researches. Thus a new area of research usually emerges. With reference to this thesis, dynamic non-stationary financial processes with fractional components in their structure will be considered as a new research area. The real financial market is characterised with "statistical fractionality", and non-stationary fractional differential equations in partial derivatives, where a time series of spectral parameter $q(t)$ is chosen as the main research object for the theoretical and experimental estimation in this thesis.

3.2 BASIC PHYSICAL APPROACHES TO THE STUDY OF PATTERNS OF BEHAVIOUR IN OBSERVED VARIABLES

The classical approaches when studying the behaviour of simple single objects have been examined by [7, 23, 28, 32, 33, 34]. These approaches were founded by Isaac Newton, who proposed describing various events with differential equations and dynamic systems. With rare exceptions, all modern theories are now established in this language of mathematical physics [32, 33, 34]. Differential equations have become the basic tools for mathematical modelling and partial differential equations are considered as a generalisation of differential equations in infinite phase space [7]. As a result, the mathematical approach was transformed into functional analysis [35]. However, there are many problems, for which it is impossible to write integral equations, where position or speed is less important than type of behaviour. Therefore there was a need for a qualitative theory of differential equations, whose basis was founded by Poincare. The main objective in this theory was finding characteristics of the differential equations, rather than their solution [7, 23, 36]. Thus, instead of finding solutions (in the form of a function), the main focus of a qualitative theory of differential equations moved to *phase space*, where points depict system states, and its transformations, which determine dynamic equations. In other words, the qualitative theory of differential equations leads to a geometric interpretation of the motion equations as a single-parameter (time dependent) family of the phase space transformations. The term *attractor* defines the attractive set in phase space. Thus, it became possible to split the behaviour of a dynamic system into two stages: (a) *transitional behaviour* (when the trajectory tends to the attractor), and (b) *asymptotic behaviour* (when the trajectory coincides with an attractor or is close to it). This allows for two types of systems – *conservative* and *dissipative* [7].

Being a basis of non-linear dynamics, the geometric approach allows for the forecasting of new effects and phenomena. Among the key and paradoxical non-linearity phenomena is chaotic state, or *dynamic chaos*. This has changed the perception of the simulation of natural phenomena and the ability to analyse and forecast [7, 9,..., 12, 19, 21,..., 28, 36]. Indeed, the classical theorem of continuity and differentiability of solutions of differential equations (with initial conditions) justified the use of deterministic methods of solutions. However, formal reasoning, made on the grounds of this theory, lead us to an erroneous belief, that analysis, description and forecasting of the behaviour of any deterministic system impose no limitations. Furthermore, this allows one to distinguish two types of dynamic system – *deterministic* and *stochastic*. For deterministic systems, it is possible to give the analysis or/and forecast for any period of time, but for stochastic systems it is not possible to talk about a known analysis or/and forecast and it is only possible to deal with stochastic characteristics. There is another important type of intermediate non-linear systems, which nominally can be considered as deterministic because knowing the present state it is possible to find what is going to happen to the system at any future point of time. But the forecast is only available for a limited period of time, as minor inaccuracies in the initial state of such a system accumulates with time, making long term forecasts impossible. These systems start to behave chaotically, and again it is only possible to talk about a statistical analysis and description.

A great contribution to the study of dynamic chaos was made by Smale, who proposed the representation, known as "Smale's horseshoe" [7]. The author showed that the time dependent behaviour of a dynamic system does not follow the motion of a point over a torus, or over any other non-convex surface, but follows the motion over an "indefinite disrupted" surface. Furthermore, this system could be structurally stable and it is not feasible to make changes by correcting the system even slightly. In subsequent studies these objects were defined as *fractal sets* or *fractals* [37, 38, 39]. Among fractals, the main attention is usually paid to random fractals, which are constructed on the basis of initial randomicity (resulting from random processes taking place in the system considered), rather than on a random disturbance within a deterministic fractal [37]; and to multifractals, which can not be characterised with a single parameter (e.g. fractional dimension), but with a function – the spectrum of dimensions [38].

Behaviour of a simple chaotic dynamic system over time may turn out to be so complex that from some points of view it could not be distinguished from a clearly random process. Thus, these systems can synthesise global stability (the trajectory does not leave some specified area of the phase space) with local instability (minor inaccuracy of the initial states accumulate, and close trajectories split). At the same time, continuous dependence and differentiability by initial conditions remains for any finite time interval, i.e. there is no contradiction with classical theorems. However, this dependence becomes "jagged" and analysis and forecasting with these functions is available only for relatively small periods of time, i.e. while the derivative is not too large.

The conceptual differences in approaches to the examination of dynamic systems must be considered. Non-linear dynamics examines the asymptotic behaviour of the system, with time tending to infinity. Once the importance of this situation was recognised, researchers have thoroughly reconsidered their approaches to random and deterministic processes. New terms, such as *dynamic chaos*, and *strange attractor* have emerged. Conceptual methods of non-linear dynamics look quite attractive for studying financial and economic processes [6, 16, 17]. However, there are only a limited number of real systems (with a limited number of degrees of freedom) which can be efficiently described with non-linear dynamic methods [7]. When the complex behaviour of simple systems with a relatively small number of degrees of freedom is barely distinguishable from random processes, it is impossible to describe the macrosystems with a large number of degrees of freedom.

Along with methods for studying dynamic systems behaviour, three other basic approaches have been formulated from physics [19,..., 28, 32,..., 35, 40,..., 44]: an approach based on the so-called molecular dynamics method; a phenomenological approach; and an approach based on the use of methods for statistical physics.

In the first approach [40, 41] it is only necessary to obtain the dependence of generalised coordinates of the macrosystem over time. One of the justifications for the use of this approach comes from *synergy*, a non-linear science [7, 10, 11, 12, 16, 17, 19, 21,..., 28]. This justification is based on the ability of a macrosystem for self-organisation during its evolution, while some basic degrees of freedom are selected from the whole set and the rest follows the selected ones. These basic degrees of freedom determine the behaviour of the system's elements at the macro level and are usually called the *order parameters*. The key role in self-organisation of macrosystems is played not only by irreversible or dissipative processes, but also by principles of self-similarity, invariance and symmetry [7, 10, 11, 12, 22,..., 28, 36, 37, 38, 42], typical of both closed and open-ended systems. However, it is still necessary to obtain information on the macrosystem elements' behaviour and define a solution to the system of dynamical equations. This system, that describes changes in elements of the macrosystem, is generally represented by a system of non-linear partial differential equations [32, 33, 41]. In general it is necessary to set precise values for all generalised coordinates at the start, which is in fact almost impossible. Even if it was feasible, then it generally would be still impossible to solve this system of differential equations for most practical cases, even on the most powerful computers.

The phenomenological approach [41, 43] is simpler because it does not require the dependencies of generalised coordinates over time to describe the behaviour of the observable variables. In other words, the phenomenological description of a macrosystem behaviour is not based on information regarding the behaviour of each element of the macrosystem. Instead of this both correlation between observable variables, obtained from experimental data handling, and the general laws of physics (e.g. the energy conservation law, the phenomenological law of thermodynamics, etc.) are used. To obtain the unknown observable variables heuristic methods can be applied, i.e. methods of plausible arguments that in particular use intuitive ideas, various analogies, etc. This way of finding the correlation between observable variables is applied when it is hard to obtain this correlation from experimental data handling. Most aspects of non-linear dynamics have implicitly affected the phenomenological approach. For instance, some of the empirical criteria dependences, widely applied in thermodynamics, theory of heat exchange and other adjacent fields, have fractal dimensions [19, 21, 22, 32, 33, 38, 43]. Interestingly, the majority of these dependences were obtained long before fractal structures were introduced. The major drawback of the phenomenological approach lies in its inability to prove the validity of the correlation between observable variables. Empirical correlations are valid only for that range of physical variables for which they were obtained that is why they have a rather limited area of application. A similar problem arises during application of the phenomenological correlations to more complex processes, for instance, it occurs while using the ideal-gas equations [33, 41]. In terms of the phenomenological approach it is impossible to state equations describing the dependence of their coefficients from parameters that characterise system elements and their interaction.

The problems previously specified can be resolved with a statistical approach [7, 9, 22, 24, 25, 27, 28, 32, 33, 36, 37, 38, 41, 44, 45]. The advantage of this approach is explained by the fact that it is based on the use of information on the behaviour of system elements over time, i.e. it has the same advantage as a molecular dynamics method. However, if the latter can only be applied to systems containing a relatively small number of elements, the former can be used for analysing macrosystems with a large number of elements. This happens because in terms of the statistical approach it is possible to find the laws of behaviour for observable variables using (unlike the molecular dynamics method) not only the behaviour information of the system, but also the information of micro and macro levels. Such systems may be chaotic at the micro level and follow a more complex dynamic behaviour at the macro level, but the average values of the observed parameters are deterministic. In other words, chaos at different levels can lead to an ordering of the whole macrosystem. A good example of such a system is a turbulent flow in water pipes [7, 19, 22, 28, 33, 36, 38, 43]. In this system self-organisation processes exist at both micro and macro levels. It is also possible to consider transport phenomena (diffusion, viscosity and thermal conductivity) [24, 27, 28, 32, 33, 37, 38, 41, 44]. These different events have a similar implementation mechanism at the micro level, which is described by the same kind of differential equations at the macro level [32]. The main attention is paid to diffusion, which is a random walk process resulting in extremely complicated particle motion. One can decide that this motion is absolutely unsystematic, nevertheless mean values are quite organised. However, this organisation may look "strange" [21] and produce rather intricate structures. Here, on the one hand, we deal with processes that occur in space and time, and, on the other hand, have another parameter of structural origin [10]. It is worth mentioning that the statistical approach has drawbacks that are mainly connected with the limits of applicability. Suffice to say that in this method the relative error for finding variables is proportional to $1/\sqrt{N}$, where N is the number of elements in the system. It is clear that when examining macrosystems, consisting of a large number of elements, this error is minimised.

As a result, in practice, the approaches discussed have been applied to physical objects [7, 9,..., 12, 19,...28, 32,..., 45]. Taking into account financial relations, and following the preceding analysis, it is worth combining methods of statistical physics with methods of non-linear dynamics, including the application of methods of statistical fractionality for this research.

Other methods will be applied when required for particular local tasks. It is also worth mentioning that physical and financial objects have fundamental differences as well as similarities. Often in physical systems it is possible to examine objects while remaining outside the system. To do this, we have to keep external factors constant and repeat the experiment many times under different conditions. In financial research it is almost impossible to reproduce external factors and influences accurately, and to stay outside the system. This has had a deep effect on the approaches used for solving problems of financial analysis, forecasting and simulation.

3.3 EVOLUTION THEORIES OF PRICE CHARACTERISTICS

Modern financial theory emerged in the mid-1920s, where it was mostly concerned with governance, administration and increasing in assets. In its development financial theory splits into two main directions. The first was founded on the assumption that price characteristics, supply, demand, etc., are totally predetermined and the other was based on uncertainty.

The key role in the development of the first theory was played by Fischer [63], Modigliani and Miller [64, 65], who were analysing issues of optimal solutions for natural and juridical objects. Mathematically, they were maximising multivariable functions with some constraints. With application to this thesis, this direction is of secondary importance, and will not be thoroughly examined here. In the second direction it is worth mentioning the works of Markovitz [66], who founded the investment portfolio theory. This theory was aimed at the problem of optimisation of investment decisions in uncertainty. Likelihood analysis, performed by Markovitz established the importance of *covariance* between prices, and stressed the importance of those components that effect the riskiness of a stock portfolio. Financial theory distinguishes two types of risk: unsystematic risk, which can be diversified and minimised (i.e. investor can affect this risk by his/her actions); and systematic risk (i.e. market risk). Markovitz [66] stated a clear perception of diversification in organising a portfolio and minimising unsystematic risk. Diversification has influenced the development of another two theories: *CAPM* – Capital Asset Pricing Model [67]; and *APT* – Arbitrage Pricing Theory [68]. These two theories have provided explanations as to how revenues on shares are calculated, which factors affect revenues, and which ideas and theories to use for estimation and calculations. Almost at the same time, new advanced systems of statistical data collection, analysis and market price forecasting were being developed to control possible system and global risks [6]. For these purposes, financial parameters and transaction hedging, that account for probability changes in future prices, were developed to reduce the risk of possible adverse effects of future price changes [6, 13].

The work of Kendall [69] has to be emphasised as the one, which led to the establishment of *Efficient Capital Market* (*ECM*) *Theory* with its applications. By analysing real statistical data, the author was unable to determine any trends, cycles or rhythms. Moreover, he was able to conclude that price logarithms of $S = (S_n)$ are random walks, i.e. if $h_n = \ln(S_n / S_{n-1})$, then $S_n = S_0 e^{H_n}$, $n \geq 1$, where H_n is the sum of independent random values $h_1, \ldots, h_n$, which can be interpreted as "logarithmic gain", "return", "reimbursement", etc.

The idea of using random walks to describe price evolution has been proposed previously, for example, in the work of Bachelier [70]. However, unlike Kendall, Bachelier believed the exact prices $S = (S_t)$, $t \geq 0$ (not their logarithms) have zero mean and fluctuations of order $\sqrt{\Delta}$, and then according to the formula $S_t = S_0 + \xi_\Delta + \xi_{2\Delta} + \ldots + \xi_{n\Delta}$ with time $t = 0, \Delta, 2\Delta, \ldots, \mathrm{n}\Delta$, where $\xi_{\mathrm{n}\Delta}$ are random independent identically distributed (IID) variables equal to $\pm\sigma\sqrt{\Delta}$ with probability ½ . After Bachelier made the formal limited transformation of $\Delta \to 0$, he found that the limit process $S = (S_t)$, $t \geq 0$ gives $S_t = S_0 + \sigma B_t$, where $B = (B_t)$, $t \geq 0$ is Brownian motion. Formally, it is called classical Brownian motion or Weiner process, i.e. random process with Gaussian (Normal) homogeneous independent increments and continuous trajectories [71, 72]. Based upon classical Brownian motion, Bachelier developed a formula for finding a rational price (premium) of a standard call-option. This formula was a precursor to the famous Black – Merton – Scholes [73, 74] formula for rational price calculation of the standard call-option, where $S = (S_t)$, $t \geq 0$ is described with geometrical or economic Brownian motion [75].

Before Kendall there were already a large number of publications on empirical analysis of financial characteristics, aimed at answering the question how to forecast prices, price movements, etc. Among them were the works of Cowles [76, 77], Working [78], and Cowles and Jones [79]. These works supported the findings that increments $h_n = \ln(S_n / S_{n-1})$ of price logarithms S_n, $n \geq 1$ are more likely to be independent, and that the sequence $H_n = h_1 + h_2 + \ldots + h_n$ is a random walk process. Kendall's work [69] stimulated in-depth analysis of financial indicators' dynamics and the development of probabilistic models, explaining observable effects. Moreover, these issues are still of great interest, since the problem of the correct description of the dynamics of price characteristics $S = (S_t)_{t \geq 0}$ is far from solved, and is still the subject of modern theoretical, probabilistic, statistical and dynamic investigations.

The adaptation of the random walk hypothesis for describing the evolution of price characteristics has led to the classical ideas of the ECM theory. These ideas were initially focused on finding supportive arguments for using a probabilistic approach, and thus justifying the employment of the random walk and more the general martingale hypotheses [4, 5, 6, 8, 9, 13, 18, 71, 72]. Each of these hypotheses has determined the using of different mathematical techniques of the stochastic calculus theory of natural sciences [7, 9,..., 12, 15, 19,..., 28, 37, 38, 41, 45]. Here it is also worth referring to the works of Roberts [80] and Osborne [81]. Roberts' [80] study that follows Working and Kendall, who address the practices of the financial market and provide heuristic considerations to the random walk hypothesis. Osborne [81] also came to the same conclusions that price logarithms (not prices) follow Brownian motion with drift, when examining real share prices. The finding was also supported by Samuelson [75], who introduced the previously mentioned geometrical (economic) Brownian motion in the form of $S_t = S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t}$, $t \geq 0$, where $B = (B_t)$, $t \geq 0$ is standard Brownian motion.

The idea of an ECM is based upon the assumption that price characteristics are momentarily affected by any new market information, and react in the way which does not allow one to buy at a cheaper rate without any risk and sell at higher prices, i.e. does not allow for arbitrage. The term "effectiveness" describes that the market rationally reacting to all changes in information, and:

- momentarily corrects prices, which always reach an equilibrium that is "fair", thus leaving no room for gaining profit due to price differences without risks;
- incoming information is uniquely and adequately interpreted by market participants, who momentarily correct their decisions;
- market participants are "collectively-rational" and similar in terms of the objectives.

Formally, the effectiveness can be viewed as both related to and resulted from the relevant information. Here, it is possible to distinguish three types of available information: information coming from previous prices; publicly available information; and all the other information available. All previous concepts can be viewed as the first basic ground for the conceptual approaches to analysis and forecasting of price characteristics.

From the probability point of view, the term "information" can be considered in association that market "uncertainty" can be seen as "random" within some probabilistic space (Ω, F, P), where [71, 72]:

- $\Omega = \{w\}$ is the space of simple events (outcome space);
- F is a σ-algebra of a Ω subset; and
- P is a probability measure on (Ω, F).

If the probabilistic space is added with flow (filtration) $F = (F_n)$, $n \geq 0$, consisting of σ-algebra subsets such as $F_m \subseteq F_n \subseteq F$ when $m \leq n$, then events from F_n can be interpreted as "information" available for market participants before time n. To clarify the term "events", it is worth highlighting that in examining price characteristics, market participants are usually less interested in the outcome itself, but are more concerned with the subset of outcomes, to which it belongs. In stochastic analysis spaces $(\Omega, F, F = (F_n), P)$ with the flow $F = (F_n)$ of σ-algebras is referred to as filtration probabilistic spaces. In financial mathematics the $F = (F_n)$ flow is also recognised as information flow (traffic), which helps in determining different forms of market effectiveness, such as weak or strong [82].

Mathematically, the idea of a rational and fair market was employed within *martingales*, which (normalised) price characteristics to some measure, equivalent to a basic probability measure [4, 6, 71, 72]. The martingale hypothesis is a generalisation of the "random walk" hypothesis of ECM. Why is the martingale hypothesis natural in this case? There are several answers to this question, but the best comes from the theory of non-arbitral markets in which market effectiveness (or rationality) is associated with a lack of arbitrage, resulting in the emergence of martingales. It is worth saying that if $X = (X_n)$ is a martingale in relation to a $F = (F_n)$ flow, where $X = x_1 + x_2 + ... + x_n$, $x_0 = 0$, then $x = (x_n)$ is a martingale-difference, where x_n are F_n-measurable, $\mathrm{E}|x_n| < \infty$, $\mathrm{E}(x_n \mid F_{n-1}) = 0$, E is the mean distribution. Consequently, hypothesising that for any $\mathrm{E}|x_n|^2 < \infty$, $n \geq 0$ and $k \geq 1$, we have $\mathrm{E}x_n x_k = 0$, i.e. $x = (x_n)$ are uncorrelated. In other words, square-integrable martingales belong to the class of random sequences with orthogonal increments, where $\Delta X_n = X_n - X_{n-1} = x_n$ and $\Delta X_{n+k} = x_{n+k}$. Thus, market effectiveness can be considered simply as a martingale of asset prices. A particular case of such a market is the one where price characteristics walk randomly. A martingale *per se* is a more common stochastic process than a random walk, because for martingales all changes come from values of the random variable, which has to have zero mean and should not necessarily have constant variance. It is also important that these changes do not necessarily have to be independent [4, 6, 71, 72]. The martingale class is vast, and the martingale property $E(X_{n+1} \mid F_n) = X_n$ implies that the most it is possible to say about the increment's $\Delta X_n = X_{n+1} - X_n$ forecast (on the basis of information F_n) is that on average (in relation to F_n) this increment equals zero. This finding corresponds with the intuitive perception that in a "fair" market there is no room (with positive probability) for one to gain and others to lose. In such a market, the conditional profit $E(\Delta X_n \mid F_{n-1})$ has to be zero.

The empirical analysis of price evolution [51] shows that after some point of time (Figure 3.1) the autocorrelation of h_n goes to zero. This can also be interpreted as indirect confirmation of the martingale hypothesis.

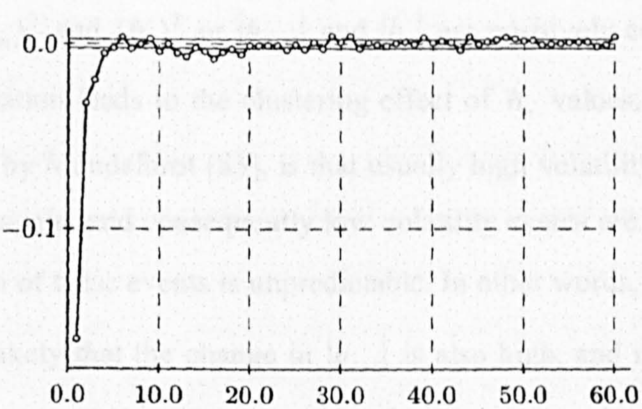


Figure 3.1. Empirical Autocorrelation Function of h_n for DEM/USD with $\Delta = 1$ Minute

The ECM hypothesis has pushed the development of new financial techniques, suitable for conservative investors that are fond of the diversification idea. Also, this hypothesis has increased the wide application of methods and approaches of mathematical simulation and analysis previously used in the natural sciences to be applied to economics and finance. Thus, it is worth in the next section paying more attention to the description and development of ECM theory.

3.4 The Development of Efficient Capital Market Theory. Market with Fractal Structure.

The idea of the ECM was employed within martingales, which describe normalised prices of this market to some measure, equivalent to some basic probability measure. Thus it is possible to say that the assumption of martingale prices corresponds to the economic assumption of a "well-developed" market, where today's best forecasted price for tomorrow, the day after tomorrow, etc. is exactly the price today (from a mean-square estimation). In other words, forecasting on the ECM is trivial and formally excludes any possibility of predicting future price movements. However, market participants still rest their hopes upon the possibility of predicting future price movements. What gives birth to their hopes? With reference to the financial analysis of empirical data, it is worth starting with some nontrivial phenomena of price movement features, like clustering (grouping), to find probabilistic statistics of price structure as random processes.

Let logarithms $H_n = \ln(S_n/S_0)$ of normalised prices and the sequence $H_n = h_1 + h_2 + ... + h_n$ be martingale with regard to filtration F_n. Then values of h_n produce the martingale-difference $\mathrm{E}(h_n \mid F_{n-1}) = 0$ and became uncorrelated $\mathrm{E}h_{n+m}h_n = 0$, $m \geq 1$, $n \geq 1$, when h_n^2 are integrated. However, it is known that uncorrelatedness does not imply independence, and does not exclude that for example $(h_{n+m})^2$ and $(h_n)^2$ or $|h_{n+m}|$ and $|h_n|$ are positively correlated. The phenomenon of positive correlation leads to the clustering effect of h_n values, $n \geq 1$. The heart of this effect, described first by Mandelbrot [83], is that usually high volatility events are followed by high volatility events again, and consequently low volatility events are followed by low volatility events, but the sign of these events is unpredictable. In other words, if the change in $|h_n|$ was high, then it is very likely that the change in $|h_{n+1}|$ is also high, and if the change in $|h_n|$ was small, then it is very likely that the change in $|h_{n+1}|$ is also small. Naturally, these features are typical of most prices and financial indices, including most currency exchange rates. Following [6, 52], Figure 3.2 depicts "clouds" of small and high values of changes in $|h_n|$.

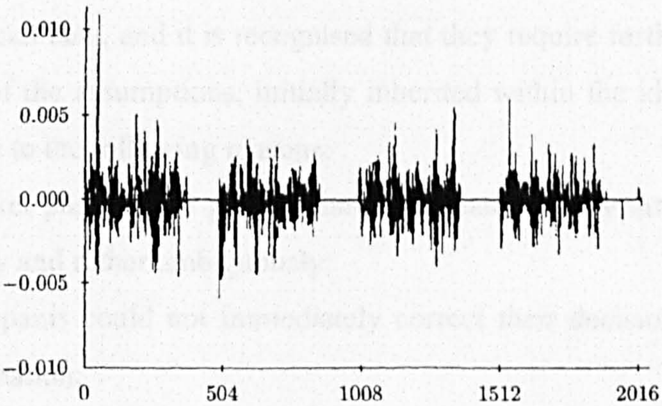


Figure 3.2. Clustering Effect of Changes in h_n for DEM/USD with $\Delta = 20$ Minutes

This gave birth to the hope for more nontrivial, and also non-linear, forecasting of (for instance) values of $|h_{n+m}|$, and of getting more detailed information on the distribution of the sequence

h_n. Also, even the simple assumption that the probabilistic nature of values $h_n = \sigma_n \varepsilon_n$, where ε_n are independent standard normally distributed variables; $\varepsilon_n \sim N(0,1)$; and σ_n are constants (which are standard deviations of h_n, i.e. $\sigma_n = +\sqrt{Dh_n}$), is not experimentally supported. Statistical analysis of many real financial data sets shows [4, 6] that the empirical PDFs of h_n are more gaunt (spear-shaped) around their mean values, and consequently the tails of the h_n distributions are "heavier" than for the normal distribution. In the financial literature the values of σ_n from $h_n = \sigma_n \varepsilon_n$ are usually referred to as *volatility*. It is also important that the volatility of most financial data is volatile itself. In other words, $\sigma = (\sigma_n)$ is not only a function of time or a constant, but also is random.

These representations enlarge the traditional class of linear Gaussian models, and introduce the new class of non-linear conditional Gaussian models, where h_n is a mix of Normal distributions, averaged on a volatility distribution σ_n^2 ("random variance"). Mathematical statistics is also familiar with the fact that a mixture of distributions with quickly decreasing tails may result in distributions with heavy tails. As well as the clustering effect there are other empirical phenomena, showing that the link between price and volatility is more sensitive. It is known, when volatility is small, then prices usually tend to fall or rise for as long as possible. Similarly, if volatility is high, then prices tend to have inverse price trends.

Thus, it becomes clear that the classical model of a Gaussian random walk $h_n = \sigma_n \varepsilon_n$, which was described above, and the ideas of the ECM do not entirely reflect the dynamics of the changes in real financial data, and it is recognised that they require further clarification and adjustment [6]. Many of the assumptions, initially inherited within the idea of ECM, raise quite justified criticism due to the following reasons:

- although market participants get the same information, they interpret it and react to it nonuniformely and rather ambiguously;
- market participants could not immediately correct their decisions according to the incoming information;
- attitude of market participants towards risk can be quite subjective and people's decisions appear to be non-linear;
- objectives of the participants may vary. For example, time intervals of traders' financial activities differ, being very short for "speculators" and long for central banks;

- the idea that ECM is martingale in relation to some probabilistic measure or information flow allows for an exact mathematical expression only if it is based on the hypothesis of a non-arbitrary market;
- this implies that there is a class of martingale measures, in relation to which price characteristics are martingales. Meaning that instead of one, there could be a set of stable states of the market, which indicate different objectives for the market participants;
- the market can be either in a position at or close to equilibrium, or it can be a great distance from equilibrium, so a market's perceptions may be inadequate in relation to the information obtained;
- heterogeneous reactions of numerous markets' agents with different time horizons can generate new events, resulting in secondary reactions of market participants.

The fact that the market is occupied with investors having different interests and abilities is more positive than negative, since this makes the market increasingly diversified and liquid, and thus more stable. According to many scholars [59], a market becomes unstable when long-term investors quit and try to sell their assets and become short-term investors. All this implies that heterogeneity, multiplicity and the fractional structure of interests of market participants are needed for a stable market. The fact that the market is *statistically fractional* was firstly noted by Mandelbrot in the 1960s and from then there has been an increasing interest in the subject detecting statistical fractional structure in currency exchange rates, short-term statistical fractional structure in bonds and shares prices, etc.

Similarly to the description of the ECM above and following [6], the main features of a market with fractional structure (or fractional market according to [59]) are:

- investors adjust prices according to relevant information for their time horizon at any point of time; the reaction can not be immediate, but will take effect only after being confirmed;
- technical information is the determining factor for short time horizons, while for longer time horizons fundamental information is essential;
- prices result from actions of both long-term and short-term investors;
- high frequency price components are determined by "short-term" investors, and low frequency (more smooth) components are determined by "long-term" investors;
- liquidity and stability fail when investors with different time horizons leave a market, i.e. when the fractional structure of a market disappears.

Thus, the ideas of efficiency, no arbitrage and fractional structure complement each other. For instance, many non-arbitrary models have a fractional structure; and fractional processes can be martingales, then, the corresponding market is non-arbitrary, but could be non-martingale too, like fractional Brownian motion [6, 37]. As a result, following qualitative description of fractional markets, it is quite likely that the FX-market belongs to the type of markets open for arbitrage. The simplest examples of such models are modified models of Bachelier and Black-Merton-Scholes, where Brownian motion is replaced with fractional Brownian motion with $H \neq 1/2$ [84], H – the Hurst parameter. Note, fractional Brownian motion with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ in not semimartingale [4, 6] and does not have martingale measures. This circumstance gives a chance (but not necessarily) for arbitrage in such models.

It is possible to conclude with current knowledge that it is too early to talk about an economical and mathematical theory of the financial market and its foreign exchange sector as a "unified complex dynamic system", working for a real environment on the FX-market rather than under "classical" equilibrium conditions. Today scholars only gain facts and accumulate experience, clarify models and search for new indicators, thoroughly describing real financial environment. The above discussion shows that the direct transfer of approaches and techniques applied in the fields of science to financial or economic objects is not appropriate, although it is possible to apply more general physical approaches to methods of mathematical simulation of economic and financial processes. These approaches can be useful not only for the classification of mathematical models and for checking the adequacy of these models, and developing necessary theoretical grounds for comparing results, which have been obtained with their help, but also for the identification of the key problems, arising during the analysis of economical and financial time series.

3.5 GENERALISED CLASSIFICATION FOR MATHEMATICAL MODELS OF ECONOMIC AND FINANCIAL PROCESSES

Various mathematical methods have been applied to formulate economic and financial models [4, 5, 6, 8, 9, 13, 14, 16, 17, 18, 46]. It is clear that mathematical methods are often inappropriate for direct investigation of financial or economic objects, although they can be used for the description of corresponding mathematical models. As a result, the ideas of *mathematical methods* and *mathematical models* have become interrelated in economics and finance [14]. Thus one of the main objects of this research is the use of mathematical models, which allow us to correctly handle and interpret experimental results, and analyse and describe the processes. In general financial and economic tasks and problems can be divided into theoretical and applied tasks [18]. Theoretical tasks consider the most common features of a system and their components, and allow subsequent conclusions to be drawn from formal feature analysis. Applied tasks attempt to estimate the parameters of specific objects and prove conclusions and forecasts to validate these specific decisions. In this research it is worth looking at the analysis of applied tasks within financial analysis. So for this, statistical time series data on currency exchange rates is empirically analysed.

With relation to the mathematical models considered above, economical and financial tasks and problems are quite similar – both internally and in terms of optimisation and managerial decision-making. Therefore following [4,..., 9, 13, 14, 16, 17, 18], it is worth classifying mathematical models in order to find the limits of applicability of these approaches, which are used for analysis and forecasting the observable parameters. Figure 3.3 presents a general classification of mathematical models.

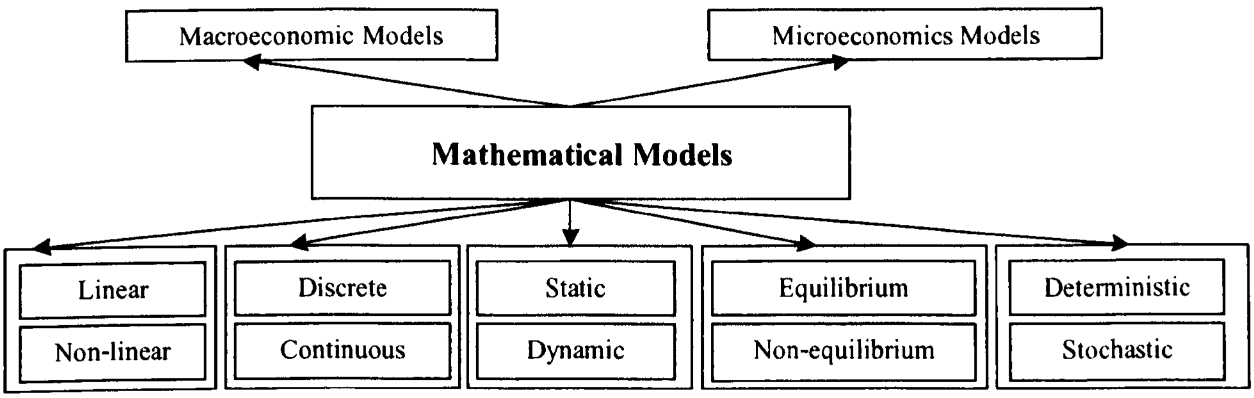


Figure 3.3. General Classification of Mathematical Models

This classification is neither unique nor thorough for analysing and handling data on exchange rates dynamics. Moreover, being rather conditional and dual in character, it nevertheless stresses the necessity of examining some key characteristics, problems and approaches. There are great differences among exchange rates and other financial indicators such as indexes, share prices, stock, etc. For example, if buying or selling shares is associated with investments in corresponding fields, then buying or selling of currency tends to provide a basis for production, consumption, etc. Also, when exchanging outstanding amounts of currency, at least two countries are affected. The exchange rate depends on the macroeconomic and political situation, and is also heavily influenced by central banks' policies (interest rates, interventions, etc.). These factors affect both exchange rate dynamics and behaviour, and the specifics of applied models.

In terms of applied time series analysis for the foreign exchange market, the main attention has to be paid to microeconomic models, and to "supply-demand" models in particular. However, results obtained from these models (especially for long time horizons), have to be preferably viewed as constituents of macroeconomic models.

For dynamic models, the problem of distinguishing stationary and non-stationary objects has to be emphasised, because it is important to know in advance for how long we will be examining the same dynamic system. The same problem affects the choice of precise mathematical model, since it is recognised that not all of the financial time series are stationary. As this work is a computer-based research, discrete models are taken as the key models for the proposed study, and the type of available experimental data, is evidently discrete, also confirms this choice. The discreteness of experimental data is closely connected with the processes of financial analysis and simulation. In particular, changes of financial data in empirical analysis shows that during the 1970s and before, researchers were operating with rather wide time intervals – years, quarters, months, weeks. The most typical probabilistic models at that time were random walks, autoregressions and moving average models, and their combinations. It was important that all models applied were linear, and daily financial data was sufficient. Following the analysis of daily data, numerous modifications of non-linear models emerged during the 1980s. For their practical realisation intraday data were required, usually with hour discreteness. During the 1990s it also becomes possible to perform full-scale intraday data analysis. These possibilities have resulted from the evolution of IT and computing, which has significantly increased the effectiveness of obtaining and handling of information now with minute and even second discreteness. As was already noted, the main impact on foreign exchange is from the long-term investors, and their activities are a low frequency in price characteristics. This thesis considers intraday analysis of price characteristics only as supplementary.

Generally, time series are defined as a sequence of observed variables or characteristics, put in chronological order. For currency exchange rates we have only subclasses of such sequences, or more precisely those of them, that could be related to observations fluctuating by nature. In other words, we postulate that we do not have purely deterministic schemes of dynamic observations, where elements of sequence can be unambiguously calculated only as values of some non-random parameter. The main attention will be paid to the fluctuating component Z in equation (2.1), and then to the cyclic variable Y. The analysis of the trend parameter X of the observed empirical data S will be performed only if X determines the non-stationary nature of S. That implies that *per se* we will consider dynamic systems with noise of different natures, paying attention to noise rather than dynamics. Some important problems usually emerge while analysing such systems. Firstly, they come from uncertainty which mathematical model has to be associated with the results obtained, so the choice between equilibrium and non-equilibrium, determined and/or stochastic models is nontrivial. Studying and comparing of fractional structures turns out to be necessary for understanding which models of evolution of price characteristics are most valid; why stable financial systems have to have fractional structure; and for determining other issues, related to non-linear systems. Thus, particularly the analysis of the justified and correct choice of models is one of the focuses of this research.

A successful mathematical simulation is based on the sequence *model – algorithm – software* [23], and mathematics acts not only as a tool, but also as a source of new ideas, so it is worth examining the basic models in more details. Starting with the analysis of relatively simple linear Gaussian stationary models, we will then turn to those models, which describe phenomena, which appear during empirical currency exchange rates investigations, such as the clustering effect.

3.5.1 Linear Mathematical Models

As is well known, we can only use linear equations [23, 32] to solve various problems of natural science. These are the problems, which either accept relatively low accuracy, or where the impact on the considered system is relatively low, and reliable analysis and/or forecast is required for a relatively narrow range of changes of the observed parameters. Linear equations are suitable for many practical applications, and the superposition principle allows us to find solutions to more complex problems on the basis of relatively simple ones. A lot of methods, which focus on the use of linear equations, have been developed [23, 32, 41], and important non-linear equations can be simplified to linear equations [23, 33, 41, 43, 44, 45] as well as analysis of stability of solution of non-linear problems being reduced to linear equations [23]. It is known [33] that most, but not all, correlations between variables that describe some natural phenomena are linear under certain conditions. In other words, the role of linear equations in the development of modern science and technology is very important, but not unlimited.

We now consider linear models known and applied within finance and economics. The key emphasis will be placed on the fluctuating component (in particular on a stochastic one) for analysing financial processes [4,..., 8]. Therefore, the area of stochastic linear models is quite vast and many publications are devoted to it, e.g. [4,..., 8, 11, 85,..., 89]. In general time series theory there is a set of standard linear models, for example: Autoregressive (AR); Moving Average (MA); Autoregressive Moving Average (ARMA); and Autoregressive Integrated Moving Average (ARIMA). We will not consider them in full detail, but will introduce their main application characteristics.

AR, MA and ARMA models focus on the analysis of stationary time series. These models are able to describe the existing relations between variables in stationary time series, and an attempt to apply these models in a pure form to non-stationary time series could lead to false conclusions. Therefore, in their application special attention is paid to the process of the identification of the time series and models, which is developed in econometrics [4, 5]. Real time series in finance are mostly non-stationary, and the use of these models can only be efficient for a short-term forecast. Non-stationarity of time series could arise from existence of a non-random component depending on time. So it is possible to talk about non-stationary homogeneous time series. ARIMA model and its modifications has been successfully applied to describe the behaviour of non-stationary but homogeneous time series, and in particular time series with a cyclic (seasonal) component. It is worth noting that noise is a necessary component of these models. Models like ARIMA help to solve problems for short-term and medium-term analysis and forecasting. Most efficient forecasts are made with the use of identified (fitted) models, such as ARIMA (p,d,q). In its framework, the so-called adapted methods, which allow one to find or/and make forecasts that can be updated with the help of relatively simple mathematical procedures when new data is added. Sometimes models like ARIMA are also called linear digital filters, because a process of generation of time series using this system is the filtration process of uncorrelated noise.

When setting up digital filters, problems have to be formulated differently – the coefficients in the models have to ensure the necessary spectral properties of the output signal. The problem of identification turns into the problem of time series transformation. To illustrate this, let us provide the following example. Since we only know time series x_i, and it is always possible to construct noise as a sequence of uncorrelated and identically distributed random variables ξ_i with zero mean ($\mathrm{E}\xi_i = 0$), then presenting the i^{th} element from the series x_i as some function m of previous elements $x_{i-1},\dots,\ x_{i-m}$ and $k+1$ of random variables $\xi_i,\dots,\ \xi_{i-k}$ in the form $x_i = F(x_{i-1},\dots,\ x_{i-m}, \xi_i,\dots,\ \xi_{i-k})$ we limit ourselves to a linear functions, i.e. with models like: $x_i = a_0 + \sum_{j=1}^{m} a_i x_{i-j} + \sum_{j=0}^{k} b_i \xi_{i-j}$. This is a discrete analogue of a convolution of two signals $x(t)$ and $\xi(t)$ with a finite (unequal to zero during the final duration [0, T] only) functions $a(t)$ and $b(t)$: $x(t) = \int_0^T a(\tau)x(t-\tau)d\tau + \int_0^T b(\tau)\xi(t-\tau)d\tau$. Since the Fourier Transform $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$ of the convolution $x(t) = \int_{-\infty}^{\infty} a(\tau)y(t-\tau)dt$ equals a multiplication of Fourier transforms $X(\omega) = A(\omega)Y(\omega)$, then: $X(\omega) = A(\omega)X(\omega) + B(\omega)\Xi(\omega)$; $X(\omega) = \dfrac{B(\omega)\Xi(\omega)}{1-A(\omega)}$. In this case, fitting functions a and b transform a noise spectrum into a similar spectrum of the input signal.

These correlations exist for discrete signals too. Suppose that x_i and ξ_i are periodic series with period N, $i = 0,\dots,N-1$. Then it is possible to use a cyclic change of indices for the corresponding summations in models, i.e. to use $N-1$ instead of (–1), $N-2$ instead of (–2), etc., and similarly write 0 instead of N, write $(+1)$ instead of $N+1$, etc. This summation with cyclically changing indices is usually referred to as cyclic discrete convolution. Accordingly, the discrete Fourier transform (DFT) of a sequence is [37]: $X_k = DFT(\{x_i\})k = \sum_{i=0}^{N-1} x_i \exp\left(-j\frac{2\pi ik}{N}\right)$, where $k = 0,\dots,N-1$, j – imaginary unit, $j^2 = -1$. For quick calculation of these sums the fast Fourier transform (FFT) can be used. However, it does not work for any N. FFT requiring $N = 2^m$. If a sequence x_i satisfies the requirements of the model being considered, it is possible to say that the DFT of x_i and ξ_i (X_k and Ξ_k correspondingly) are connected via $X_k = C_k\Xi_k$, where C_k is digital filter description, independent of ξ_i. Thus, using the DTF and/or FFT it is possible to make a quick convolution. In general, digital signal handling is a separate interesting field, which is not examined here in full detail. For more information please refer to [7, 37, 90].

From the point of practical use of linear models, it is possible to highlight two reasons, which makes them so popular. Firstly, they can allow analytical results to be obtained, and, secondly, their development and use takes relatively small PC resources. Thus linear mathematical models can be considered as not only useful auxiliaries, but also as one of the key tools for time series transformation. This choice is justified for analysis and short-term forecasting of relatively stable sectors of the financial market, staying in conditions close to equilibrium, and with stationarity and/or homogeneity within the given time series. Despite a variety of considered objects and techniques, a similarity in approaches to linear stochastic models in technical sciences and financial analyses is obvious. Basic forms of statistical and phenomenological approaches are used in linear models of financial analysis. Thus, not only the advantages, but also the limitations are obvious. These limits are caused by the presence of short-memory, by assumptions about stationarity and/or homogeneity, proximity to equilibrium, etc. If for the majority of cases in the technical sphere these limitations are mostly minor and acceptable, then for financial analysis and forecasting of price characteristics these limitations are crucial. Thus, following the necessity to overcome limitations of linear models, researchers have turned to studies of new approaches and techniques for analyses.

3.5.2 Non-Linear Mathematical Models

Linear dynamic models are unable to explain many observable phenomena. So, these models can either describe an outburst or some equilibrium and/or stationary status, including simple fluctuations or full-stopped movement [16, 32]. At the same time, they do not determine reasons that can destabilise the system. These models are basically unable to specify the limits of applicability and can badly distort the boundaries of approximation intervals. It is possible to get out of the restrictions by switching to non-linear models. In engineering non-linear models allow for the description of the processes in a wider range of parameters change [24, 32, 33, 41, 44]. Thus, non-linearity changes not only the quantitative characteristics of the processes but also their qualitative side [10, 11, 12, 19,..., 27, 32, 33]. This is important for a study of non-equilibrium, heterogeneous and non-stationary processes under highly intensive external shocks.

At the heart of non-linear models lie non-linear differential equations, usually in the form of partial derivatives, for which nowadays there is no complete theory, and a general method of problem solution has not been completely developed. Non-linear models and methods are usually divided into two types: parametric and non-parametric. For parametric models, the function $x = F(x, \xi, a)$ is the same for all values of x and ξ, and depends on several parameters a, which have to be obtained from the time series. Non-parametric models use local approximations around some set of points $\{x_k, \xi_k\}$, so that the function becomes a set of piecewise approximations around predetermined centres (usually piecewise linear). Each of these local functions also depends on some parameters, but the set of parameters is unique for a particular point, so the term "non-parametric regression" sounds strange, although it is widely applied in statistics (e.g. see [91] and references there). For a number of non-linear problems it is possible to find precise analytical solutions. Their examination allows new processes to be revealed [24, 32]. Nevertheless, numerical methods are basic tools for solving non-linear problems, and computer simulation have played a key role among these methods [7, 9, 10, 19,..., 23]. Economics and finance are not exceptions from this point of view [1,..., 6, 8, 16, 17, 59, 92], so switching to non-linear models was justified from instability within financial markets and financial risks [6, 93] and explained phenomena inaccessible to linear models [6, 59, 93, 94].

These circumstances have determined the foundation of non-linear stochastic models like Autoregressive Conditional Heteroskedastic (ARCH), Generalised Autoregressive Conditional Heteroskedastic (GARCH) and many modifications, including multiple-factor modifications [59, 92, 93]. They all use a class of conditional Gaussian models for discrete time [6]. Each of these models will not be considered in full details, and only a relevant discussion of their abilities will be provided. It is supposed that the variation in data remains stationary throughout linear models, and if this is not the case, then the model is said to suffer from heteroskedasticity. This is prevalent within financial parameters whose variance change over time [4]. The basic ARCH model was developed by Engle [95] to describe changes of variance with time. Detailed analysis of ARCH(p) models and their practical applications is presented in [4, 6, 59], and here we will only concentrate on explaining the clustering effect with the help of this model.

Let the conditional Gaussian model $h_n = \sigma_n \varepsilon_n$ describe the evolution of variables $h_n = (h_n)_{n\geq 1}$, where $h_n = \ln\left(\dfrac{S_n}{S_{n-1}}\right)$. Volatilities σ_n can be determined as $\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i h_{n-i}^2$ with $\alpha_0 > 0$, $\alpha_i \geq 0$ and $h_0 = h_0(w)$ – a random variable, independent from $\varepsilon = (\varepsilon_n)_{n\geq 1}$. Often, h_0 is considered as a constant or random variable with an average distribution squared, i.e. $\mathrm{E}h_n^2 < \infty$, $n \geq 0$. As a result the volatilities σ_n are predictable functions from $h_{n-1}^2,...,h_{n-p}^2$, and higher (smaller) values of h_{n-i}^2 result in higher (smaller) values of σ_n^2. If h_n^2 turns out to be high, while previous values $h_{n-1}^2,...,h_{n-p}^2$ were small, this occurs because of high values of ε_n. Thus, it becomes clear how the non-linear model can describe such effects as clustering, i.e. when values of h_n group into sets of high and sets of small values. The conditional variation (volatility) σ_n is heterogeneous, as it depends on previous values $h_{n-1}^2, h_{n-2}^2,...$.

Bollerslev [96] has generalised the ARCH model and included previous values of conditional variation to avoid the long-lag structure of ARCH(q). The success of conditional Gaussian models ARCH and GARCH has caused an avalanche of model generalisations with the purpose of explaining numerous different effects. So, as an example to explain the *leverage effect*, which was discovered by Black [6], also known as the *asymmetry effect* lying in increasing volatility after the price drops down, Nelson [97] described the EGARCH(p,q) model. However, EGARCH models are not the only ones in examining leverage effect. Another good example is the TGARCH model (a modification of the TAR models [93]).

Let us mention another phenomenon – the long-memory effect in price evolution [59]. It is possible to express the dependence of random sequence behaviour from the "past" in a number of ways. To measure this dependence, an autocorrelation function has been used [4, 6]. It is experimentally demonstrated that financial time series has a stronger correlation than any coming from the ARCH and GARCH models, and even stronger than from the MA, AR and ARMA models. In the GARCH(1,1) model the correlation decreases geometrically and the memory quickly forgets the past. It is known that a stationary time series has a long-memory or strong after-effect if its autocorrelation function hyperbolically decreases to zero. For example, this is typical for fractional Gaussian noise [37]. In [98] the HARCH(p) model was introduced, where the autocorrelation function decreases slower than in the ARCH and GARCH models. The same long-memory effect is typical in the FIGARCH models [99]. This list of non-linear stochastic models is not complete. For example, AGARCH, STARCH, NARCH models, etc. [4] can be mentioned, and also models of stochastic volatility, so-called SV models [6]. In the context of this work it is possible to say that non-linear stochastic models can be considered for revealing features of financial processes that cannot be studied within linear analysis. As a result, non-linear stochastic models can be considered as one of the basics for comparing results and give satisfactory outputs and forecasts for non-stationary processes, at least for a medium-term forecast. They can allow to analyse relatively unstable sectors of the financial market.

3.5.3 Models with Self-Similarity Properties

When considering statistical analysis of financial time series, it was noted that most of time series express statistical self-similarity, i.e. their parts are organised in a similar way as the whole time series [38]. Of course, self-similarity has to be considered here not just from the classical linear point of view, where a part is a small copy of the whole, but rather from a non-linear point, where a part is a misshaped "similar" piece of the whole. The explanation can be given in terms of the theory of statistical self-similarity, which not only provides explanations of such terms as fractional Brownian motion or fractional Gaussian noise, but also encouraged Mandelbrot to create fractal geometry. The self-similarity concept is closely connected with independent persistent random variables and processes, and also with such non-probabilistic concepts and theories, such as dynamic chaos and non-linear dynamic systems.

In 1951, the British climatologist Harold Edwin Hurst published work [100], describing the unpredicted experimental effect of yearly water content fluctuations in the Nile and other rivers. The main point of this effect is the following. Let $x_1, x_2, ..., x_n$ be yearly water levels in some part of the river within the past n years. A "good" estimate of its mean value is $\frac{1}{n} X_n$, where $X_n = \sum_{k=1}^{n} x_k$. Deviations X_k for the past k years from the empirical mean, obtained for n years, are equal to $X_k - \frac{k}{n} X_n$. Minimum and maximum deviations are $\min_{k \le n}(X_k - \frac{k}{n} X_n)$ and $\max_{k \le n}(X_k - \frac{k}{n} X_n)$. Let also the range $R_n = \max_{k \le n}(X_k - \frac{k}{n} X_n) - \min_{k \le n}(X_k - \frac{k}{n} X_n)$ define deviation of cumulative variables X_k from its mean value $\frac{k}{n} X_n$ within the past n years. Then $Q_n = R_n / S_n$, where S_n is an empirical standard deviation, are normalised parameters, providing invariant statistics to $x_k \to c(x_k + m)$, $k \ge 1$. This is often needed since mean values x_k stay usually unknown. Hurst found that for high values of n, statistics $R_n / S_n \sim cn^H$, where c is some constant and H, called the Hurst parameter, is different from ½. This result was surprising, because the expected value of H was ½, as a result of the fact that for a sequence of IID random variables $x_1, x_2, ..., x_n$ with $\mathrm{E}x_n = 0$, $\mathrm{E}x_n^2 = 1$, for a high value of n the following ratios hold: $\mathrm{E}R_n \sim \sqrt{\frac{\pi}{2}} n^{1/2}, \mathrm{D}R_n \sim \left(\frac{\pi^2}{6} - \frac{\pi}{2}\right) n$ [101]. Since asymptotically $S_n \to 1$ with probability 1, for a high value of n, values of Q_n have generally to increase proportionally to $n^{1/2}$. Why and how can the parameter H of the sequence x_n be different from ½?

To answer this question let us firstly turn to the definition of a stable random variable. According to [6], a random variable is stable if for any $n > 2$ there are positive C_n and D_n, such that: $F(X_1 + X_2 + ... + X_n) = F(C_n X + D_n)$, where $X_1, X_2, ..., X_n$ are independent copies of X. If

$D_n = 0$, then X is a strictly stable parameter. $C_n = n^{1/\alpha}$ for some $0 < \alpha \le 2$. To stress the importance of α, the term "α-stability" is often used instead of "stability". When analysing the sequence $(x_n)_{n \ge 2}$, it is worth considering the structure of the empirical distribution function $\overline{F}(x_1 + x_2 + ... + x_n)$, obtained from many samples like $(x_1, x_2, ..., x_n)$, $(x_{n+1}, x_{n+2}, ..., x_{2n})$. When x_k is a deviation from the average river water level, $\overline{F}(x_1 + x_2 + ... + x_n) = \overline{F}(n^H x_1)$, where $H > ½$. Following the above discussion, one of the explanations of $H > ½$ could be that $x_1, x_2, ..., x_n$ are *independent* and *stable* random variables with stability index $\alpha = 1/H$. This however leads us not only to non-Gaussian models of processes and distributions, but also to random processes with independent increments, among which is the class of Levy processes, covering such fundamental fields of probability theory as Brownian motion and Poisson process. Of the main interest here will be α-stable processes that are Levy processes at the same time. If a process is both α-stable $(0 < \alpha \le 2)$ and meets the self-similarity condition $F(X_{at}, t \ge 0) = F(a^H X_t)$, where $a > 0$, but is not a Levy process, then the ratio $\alpha = 1/H$ does not hold. For such processes the pairs (α, H) may be such that $\alpha < 1$ and $0 < H \le 1/\alpha$, or $\alpha \ge 1$ and $0 < H \le 1$ [102]. We will not examine stable distributions and processes for the simulation of probability models of distributions and the evolution of financial indices in this thesis in full detail. Among the works on this subject we would like to highlight those by [6, 71, 72, 102, 103]. This decision is caused by the fact that the main assumptions made for stable processes do not thoroughly coincide with the beheaver of empirical data on currency exchange rates time series. In particular, this contradicts the assumption of the independence of $x_1, x_2, ...$.

Also, there are other explanations to the behaviour of the x_n sequence. So the property $\overline{F}(x_1 + x_2 + ... + x_n) = \overline{F}(n^H x_1)$ for $H \ne ½$ can hold even for normally distributed, but dependent parameters $x_1, x_2, ..., x_n$. A stationary sequence x_n must be a sequence with a strong aftereffect. Conditions $R_n / S_n \sim cn^H$ and $\overline{F}(x_1 + x_2 + ... + x_n) = \overline{F}(n^H x_1)$, which are some type of self-similarity, hold for most financial indices, including currency exchange rates [6, 37, 38, 50, 58, 59]. So it is not surprising that the above discussion on "independence and stability" of x_n or "dependence and normality" is thoroughly applied in financial mathematics, and used for analysing the fractional structure of volatility in particular. Self-similarity characteristics are typical for numerous systems with non-linear dynamics – physical, geographical, chemical, biological, etc [6, 7, 10, 11, 12, 37, 38, 104]. Until now, we presumed that the fluctuating component Z in equation (2.1) is a random variable. However, it is well known that even absolutely simple non-linear systems like:

$$x_{n+1} = f(x_n; \lambda) \tag{3.1}$$

or

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}; \lambda), \tag{3.2}$$

where λ is a parameter, can generate sequences whose behaviour is rather similar to the behaviour of stochastic sequences [23, 37]. Such systems are interesting from different points. Firstly, from equations (3.1) and (3.2) and their special modification – a logistic system (that emerges in binary form) the fractal idea can be traced, including its properties of statistical self-similarity [105]. Secondly, behaviour of such chaotic systems allows their use for simulation of the evolution of time series, especially during a crisis, which can be characterised by rather chaotic behaviour than stochastic. Thirdly, significant interest arises in the predictability of non-linear dynamic models itself, and their ability to reveal and explain new effects on the financial market. Fourthly, the development of precise quantitative criteria for the difference between chaotic and stochastic behaviour of dynamic systems gives an opportunity to predict a financial market crisis before it happens.

There is a similarity between technical and financial approaches, so we will not present here formal definitions, models and systems for dynamic chaos [6,..., 12, 16, 17, 19, 21, 23, 36,..., 39, 42, 58, 59, 92, 105,..., 109]. Following the undertaken analysis, the main characteristic of the examined sequences is *self-similarity*. Regardless of the type of system or model *R/S Analysis* has to be considered as one of the main tool for time series research.

3.5.4 Models Based on Brownian Motion

During the simulation of random sequences $h = (h_n)$, describing the dynamics of $h_n = \ln(S_n / S_{n-1})$ the *logarithmic gain*, *return*, *reimbursement* characteristics, it is usually assumed that there is some basic random sequence $\varepsilon = (\varepsilon_n)$ that generates $h = (h_n)$. Generally, this basic sequence $\varepsilon = (\varepsilon_n)$ is considered as white Gaussian noise, which meets the requirements of constructing complex objects h_n, from simple ones. The $\varepsilon = (\varepsilon_n)$ sequence can really be considered as simple because it consists of IID random parameters with a classical Normal (Gaussian) distribution $N(0,1)$. For the development of models with continuous time and complex structure, a similar role is devoted to Brownian motion, also called the Weiner process [71, 72]. Classical Brownian motion $B=(B_t)_{t\geq 0}$ is a continuous Gaussian random process, with identically independent increments with $B_0=0$, $\mathrm{E}B_t=0$, $\mathrm{E}B_t^2=t$ and covariance function $\mathrm{E}B_sB_t=\min(s,t)$, that meets the self-similarity (self-affinity) condition $F(B_{at},t\geq 0)=F(a^{1/2}B_t,t\geq 0)$ for any $a>0$.

With its rich structure, Brownian motion can be used for the simulation of various types of random processes. For instance, it can be generalised to the level of a multivariate process $B=B^1,...,B^d$, composed of d independent classical Brownian motions. Also, the self-similarity condition implies that the $(B_{at}/\sqrt{a})_{t\geq 0}$ process is also a Brownian motion. Brownian motion is the structural basis within no-after-effect processes, i.e. no-memory-effect processes [24, 44], also called Markovian processes [41]. Among these are diffusive Markovian processes $x=(x_t)_{t\geq 0}$, that are solutions to stochastic differential equations $dx_t=a(t,x_t)dt+\sigma(t,x_t)dB_t$, integrated in such a way, that $x_t=x_0+\int_0^t a(s,x_s)ds+\int_0^t\sigma(s,x_s)dB_s$ for all $t>0$, where the last integral is a stochastic Ito integral over Brownian motion. In other words, in terms of Markovian processes Brownian motion generates an independent class of Weiner random processes, where the random process $x(t)$ is a Gaussian process with homogeneous independent increments, allowing simple multidimensional generalisation.

Ito made a great contribution to the development of the theory and methods of random walk processes [24, 45]. His aim was to simulate diffusive and diffusive-spasmodic processes with local characteristics via simulation of processes as solutions of stochastic differential equations. Here, the stochastic integral plays a key role in determining the class of continuous random processes, as the class of Ito processes [71, 72]. For financial markets, Ito processes have generated a number of models [4, 6], applied mostly to evolution of interest rates, costs of shares and bonds. For more details on the most popular interest rates models $r(t)$, representing the diffusive type of $dr(t)=a(t,r(t))dt+\sigma(t,r(t))dB_t$ models, refer to [74, 110, ..., 120].

In financial mathematics a key role is also played by geometrical (economic) Brownian motion $S=(S_t)_{t\geq 0}$, which follows from the stochastic differential equation $dS_t=S_t(a\ dt+\sigma\ dB_t)$ with coefficients $a\in R$, $\sigma>0$. With the initial condition S_0, independent of Brownian motion $B=(B_t)_{t\geq 0}$, this equation has a solution $S_t=S_0\exp(at)\exp(\sigma B_t-\sigma^2t/2)$, which shows that for $S_t = S_0 \exp(H_t)$, $H_t = (a-\sigma^2/2)t + \sigma B_t$. In this case, the process $H = (H_t)_{t\geq 0}$ is called a Brownian motion with local drift $a-\sigma^2/2$ and diffusion σ^2. This local drift characterises the average speed of change $H = (H_t)_{t\geq 0}$. The coefficient of diffusion σ^2 is referred to as a differential dispersion or volatility (in the financial literature).

The diffusive Markovian processes play an important role in many areas [6, 24, 41, 44, 45, 71, 72, 103] and the fundamental place in the theory of diffusive processes take the forward:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[a(t,x)f] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(t,x)f] \tag{3.3}$$

and the inverse:

$$-\frac{\partial f}{\partial s} = a(s,y)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2(s,y)\frac{\partial^2 f}{\partial y^2} \tag{3.4}$$

Kolmogorov parabolic differential equations. These equations describe the forward and the inverse evolution of the PDF $f(x,t;y,s)$ of the random process $x(t)$. At the same time, the coefficients a and σ in equations (3.3) and (3.4) describe the intensity of the determined and random components of the processes. Initially considered by A.D. Fokker and M. Planck in their works in physics, later equation (3.3) became referred to as the Fokker-Planck-Kolmogorov (FPK) equation [24, 45, 72]. Equations (3.3) and (3.4) link the theory of random processes with mathematical analysis, particularly with the theory of differential equations in ordinary and partial derivatives. The forward and the inverse evolutions could be considered as research objects themselves. For instance, they unambiguously determine the drift and diffusion, and can be applied to multidimensional processes [24, 45, 72], to Brownian motion (with $a(t,x) = 0$), etc.

With regards to this work, these possibilities allow us to consider the FPK equations as the main research tool, giving the possibility of thoroughly examining the nature of stochastically-determined processes, occurring on the foreign exchange market. For practical application, equations (3.3) and (3.4) have some restrictions. They describe the "slow" component of the processes [41], and assume that transition probabilities have a Normal Gaussian distribution. These circumstances are crucial for financial analysis. Indeed, the drift parameters $a(t,x)$, characterising the intensity of the systematic component of the $x = (x_t)_{t\geq 0}$ process, can be represented in the form of $a(t,x) = \overline{a(t,x)} + \tilde{a}(t,x)$, where $\overline{a(t,x)}$ is the average drift value, and $\tilde{a}(t,x)$ is a random (fluctuating/pulsating) component with zero mean. Since the pulsating component changes faster than x, and for some processes could reach up to 50% of the drift value, it can be argued that quite long time intervals are needed for a correct estimation of $a(t,x)$. At least, the order of these time intervals should be greater than the order of the intervals in which $\tilde{a}(t,x)$ changes. According to Table 2.1, the average number of currency exchange rate ticks varies from 70 to 4500 per day, which corresponds to 3 to 188 ticks per hour. In other words, diffusive-type models turn out to be incorrect for many currencies when time intervals are less than 15 minutes. A similar criticism can be addressed to values of σ, included in equations (3.3) and (3.4).

Models of financial analysis and forecasting, built on the grounds of classical Brownian motion, provide a basis for ECM theory. One of the key limiting factors for the application of these models to currency exchange market is that the conditional probability of $x(t_2)$, reaching a certain value at a given $x(t_1)$, ($t_1 < t_2$), depends only on t_1 and t_2, but not on the behaviour $x(t)$ at $t < t_1$, which in most cases conflicts with experimental data. Thus, models based on classical Brownian motion could be considered only as a first approximation to real processes, occurring on the FX-market. It is evident that in this situation it is necessary to introduce random processes with memory or after-effect. The first priority has to be given to fractional processes. Here, Brownian motion can be considered as a good model of Markovian random fractals. Examination of classical Brownian motion has revealed a number of distinctive features of this motion. In particular these include [37], non-differentiability, self-similarity, fractional dimension of realisation ($d = 1.5$), etc. As a result, this has allowed a link between the theory of fractals and the theory of random processes.

3.5.5 Fractal Mathematical Models

Among the distinctive properties of fractional objects is self-similarity, which, to a wide extent, distinguishes these objects from classical Euclidian objects. When considering the examples of self-similar objects in the field of fractional geometry, it is possible to define two quite distinctive subtypes:

- deterministic self-similarity, under which fractal consists of various fragments, identical to each other for different scales (the property of fragment scale invariance); and
- statistical self-similarity, under which fragments of fractal could change for different scales, but their statistical properties remain for any scale (statistical scale invariance).

While the existence of memory in the processes generating self-similarity objects of the first subtype is pronounced because it is initially inherited in organisational processes, the existence of memory in the processes generating objects of the second subtype is not so evident. The possibility of the existence of some memory was proved only after analysing the properties of a more general (than Brownian motion) random self-similarity fractional process, which is called fractional Brownian motion (FBM), and was investigated in work [119]. There is no simple method for FBM approximation. Mathematically, it seems most logical to use Fourier methods [120]. Some researchers apply the method of median displacement [37], however this does not lead to the real FBM. One can determine FBM with a parameter H, where $0 < H < 1$, and then a Gaussian process $x(t)$ will be FBM with transition probability:

$$\mathrm{P}(\Delta \mathrm{x} < \mathrm{x}) = \frac{1}{\sqrt{2\pi}\sigma(t_2 - t_1)^H} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}\left(\frac{u}{\sigma(t_2 - t_1)^H} \right)^2 \right) du.$$

FBM with $H = ½$ is exactly classical Brownian motion. Thus, the realisation of a one-dimensional FBM will have dimension $d = 2 - H$, and of a two-dimensional FBM: $d = 3 - H$. Unlike classical Brownian motion, in which the increments are independent, FBM with $H \neq ½$ does not have this property. However, FBM has the property of statistical self-similarity. So, if $H > ½$, then $x(t) - x(0)$ and $x(t + \Delta t) - x(t)$ are (usually) likely to have same signs, and the function $x(t)$ continues to increase if it was increasing before that. If $H < ½$, then $x(t) - x(0)$ and $x(t + \Delta t) - x(t)$ are (usually) likely to have opposite signs, and the function $x(t)$ decreases if it was increasing before that. Turbulence [19, 22] is a typical example of a system where $0 < H < ½$. This system has noise with negative covariance that corresponds to fast changes of values. Thus, many authors [6, 58, 59, 60] draw direct analogies between the properties of hydrodynamic turbulence and the behaviour of prices on the financial market. In systems with $½ < \mathrm{H} < 1$, noise has a positive covariance, long-memory and a strong after-effect [6]. This is especially important since these effects are typical of share prices, exchange rates and other financial parameters [59].

Finding dimensions for the majority of fractal curves usually requires numerous computer calculations. Nevertheless, it is easy to find the dimension $d = 2 - H$ of a FBM realisation diagram and this can be used to analyse time series [37]. Thus, in this case, the existence of memory in a time series can be estimated with the parameter H. However, H also has other implications. Here it is worth mentioning statistical *R/S Analysis* – a powerful tool for finding different effects, resulting from local and global characteristics of time series behaviour [37, 59]. Taking into account the statistical self-similarity (self-affinity) phenomena, FBM can become an explanatory techniques for these effects.

Brownian motion, as well as the Poisson process, is a random process with independent increments, that belongs to the class of Levy processes. Therefore, an examination of the features of these processes can also be considered among other problems associated with the investigation of financial processes. For example (Subsection 3.5.3), the examination of α-stable Levy processes shows [6] that the stability index (α) is connected with the parameter H through the ratio $H = 1/\alpha$. If the self-similarity condition is imposed, and relevant mathematical models are added to the ratio which has been presented, then it is possible to conclude that models with fractional dynamic, and fractal methods of financial market analysis are of the main interest in this research. A foreign exchange sector can be considered as a dynamic system, driven by self-similarity traffic.

It is possible to proceed with a review of models for financial analysis and forecasting. In fact, the variety of these models constantly increases and only one thing stays unchanged – new effects and phenomena always pop up as new research methods and tools, established in previous works in this field, emerge, which allows for us to re-consider problems from a different perspective. This usually starts new research areas and interests. With reference to this research, it is possible to consider dynamic non-stationary financial processes with fractional components in their structure as a new area. Fractional differential equations in partial derivatives can be used as a new tool. This choice is justified for the following reasons. Firstly, it logically continues the theory of financial processes and does not contradict the general trends of evolution of modern financial market. Secondly, it coincides with modern approaches in technical sciences. Thirdly, it is a natural development of the fundamental FPK equations, and is contemporary with fractal calculus. It is also worth adding that statistical fractals, including arbitrary scalable ones, are equivalent to fractional integration of white noise, what in turn is equivalent to the application of fractional differential equations. Here, q-fractional integration can be achieved through multiplication of power spectrum by f^{-2q} [38].

A number of publications have shown [121, 123] a close relationship between fractional diffusion equations of the type: $\nabla^2 p - \tau \frac{\partial^q}{\partial t^q} p = 0,\ \ 0 < q \le 1$, and $\nabla^q p - \tau \frac{\partial}{\partial t} p = 0,\ \ 0 < q \le 2$, (where p is the space-time dependent PDF, and τ is the generalised diffusivity) and continuous time random walks with either temporal or spatial invariance (fractal walks). Some authors have extended this problem further by considering fractal-based generalisation of the FPK equation [122, 123, 124]:

$$\frac{\partial^q}{\partial t^q} p(x,t) = \frac{\partial^\beta}{\partial x^\beta}[s(x)p(x,t)], \tag{3.5}$$

where s is an arbitrary function, $0 < q \leq 1$, $0 < \beta \leq 2$. Equation (3.5) is known as a fractal FPK equation. It is clear that with $q = 1$ and $\beta = 2$, equation (3.5) is the standard FPK equation. Suppose we consider a heterogeneous fractal diffusion equation of the form [122, 123]:

$$\left[\frac{\partial^2}{\partial x^2} - \tau \frac{\partial^q}{\partial t^q}\right] u(x,t) = -F(x,t), \quad 0 < q \leq 1, \tag{3.6}$$

where τ is a constant, F is the stochastic source term with some PDF, and u is stochastic field whose solution is required. When $q = 1$, this equation becomes the diffusion equation, but in general, a solution to this equation will provide temporal stationary fractal walks – random walks of fractal time.

Equations 3.5 and 3.6 can be applied to simulate non-stationary processes in financial analysis. However, they have limitations that come from the coefficient q, which is a constant. One of the ways to overcome these restrictions and to introduce non-stationarity through q, is to let it be a function of time t [122, 123, 125]. Suppose that in addition to this, the range of q was expanded to include values 0 and 2 so that $0 \leq q \leq 2$. With q varying in this range, we can either choose $q = 1$, and get the stochastic diffusion equation, or choose $q = 2$, and obtain an entirely different (stochastic) wave equation. The possibility of choosing q within this range results in control over the basic physical characteristics of the equation: it is possible to organise a static mode when $q = 0$, a diffusive mode when $q = 1$, and a propagative mode when $q = 2$. In this case, non-stationary effects are introduced through the use of a time varying fractional derivative, whose values can change the physical meaning of the equation. So, it is possible to present a non-stationary fractional differential equation in the following form [122, 123]:

$$\left[\frac{\partial^2}{\partial x^2} - \tau^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}\right] u(x,t) = -F(x,t), \quad -\infty < q(t) < +\infty, \ \forall t. \tag{3.7}$$

When $q = 0 \ \forall t$, the time dependent behaviour is determined by the source function alone; when $q = 1 \ \forall t$, u describes the diffusive process where τ is the "diffusivity" (the inverse of the coefficient of diffusion); when $q = 2$ we have the propagative process where τ is the "slowness" (the inverse of the wave speed). So, equation (3.7) is a non-stationary fractional differential equation of time dependent order $q(t)$.

Mathematical heuristics (features) of equation (3.7) are studied in full detail in [122, 123]. These works analysed the interaction between equations (3.5) - (3.7). As a result, only equation (3.7) can be proposed as the basic tool for analysis, forecasting and simulation of non-stationary financial processes. The time series $q(t)$ can be considered as an independent macroeconomic parameter, to be applied to financial analysis. However, to date there is no reliable experimental data for finding the parameters $q(t)$ for currency exchange rates. There is only criticism that this parameter exists only at the level of a postulate or an assumption [122, 123]. Moreover, there is no entirely developed methodology for finding these parameters; their practical application; implementation; interpretation; etc. Therefore, the time series $q(t)$ can be considered as the key research object for theoretical and experimental analysis in this thesis, and the task of developing and finding the evolution characteristics of the spectral parameter $q(t)$ can be viewed as a necessary and practically important problem of analysis and handling of time series of currency exchange rates.

3.6 CONCLUSION

There is a conceptual difference in approaches to analysing large and complex dynamic systems. The important role in self-organisation of such systems is played not only by irreversible, and, in particular, dissipative processes, but also by the principles of self-similarity, invariance and symmetry, therefore, the use of methods of statistical physics in combination with the conceptual approaches of non-linear dynamics, including the use of methods of statistical fractionality, seems to be justified for this thesis. There is a close correlation not only between the size of the investment horizon and the conceptual approach to the analysis and forecasting of price characteristics on the FX-market, but also among financial and physical methods applied to the description of the behaviour of the observed parameters. Together these similarities, both physical and financial objects have restrictions that affect the approaches required for solving financial problems and forecasting. Based on qualitative description of fractional markets, it is likely that the FX-market belongs exactly to this type of markets, open for arbitrage. A statistical description of a financial market is based on the concepts of efficiency, no arbitrage and fractionality, which supplement and clarify each other. In terms of the ECM-theory without abilities for arbitrage, the efficiency of the market has to be considered as a martingale price of its assets. For the real market there could be not one, but a whole spectrum of stable states. To maintain this stability, financial systems have to have fractional structure. The determining impact on the FX-market has to come mostly from the long-term investors, whose activities have low-frequency nature in terms of price characteristics.

At the moment it seems a bit early to argue on the existence of some elegant economical and mathematical theory of financial market and its foreign exchange sector, as some "large complex dynamic system", operating under real conditions rather than in "classical" equilibrium. The current status of the problem can be characterised as a period of accumulation of information, clarification of models, searching for new objective parameters, the more thoroughly describe the real conditions. From this point, the key role is played not only with the new methods of collection, storing, handling and analysing (with modern PCs/software) of real statistical data, but also with finding new methods and models, capable of providing such indicators. The terms *mathematical methods* and *mathematical models* become very closely connected in economics and finance. Successful mathematical simulation is based on a sequence *model – algorithm – software*, and mathematics acts not only as a tool, but also as a source of new ideas for simulation.

In this work, linear and non-linear models can be considered not only as supplementary tools, suitable for illustration purposes, but also as working tools for time series transformation. Models with self-similarity properties, as well as models founded on Brownian motion (including the FPK equation), have to be considered as some of the main tools for theoretical and experimental investigation for this research. New effects and phenomena are revealed after reconsidering the problems with these new research tools, based on the results of previous findings. This can results in the emergence of the new research areas. With application to this thesis, such a new research area can be represented using non-stationary dynamic financial processes with fractional components in their structure, and non-stationary fractional differential equations of time dependent order can to be considered as the new tool. So far there is no objective quantitative data, characterising the process of evolution of the spectral parameter, and there is a lack (in the literature) of well-defined methods of finding spectral parameters. Before starting practical examinations, it is first worth paying more attention to methods of analysing and handling the financial time series.

CHAPTER IV. Analysis and Handling Financial Time Series

4.1 INTRODUCTION

This chapter provides a discussion on a choice of the main experimental techniques for this research. Together with the examination of standard statistical approaches, including Ordinary Least Squares (OLS), this chapter justifies the use of the method of orthogonal regression and *R/S Analysis* as the key research methods. The use of a visual representation of the results using for example phase diagrams is described. Special emphasis is made on the estimation of the dimension of the dynamic system and its attractor, and different methods of reconstruction of the attractor from experimental data are considered. Among them, methods based on the use of the Fourier transform and spectral dimensions are argued as being of most importance. Finally, probabilistic dimensions, finding the correlation dimension and correlation integral, that plays an important role in distinguishing chaotic and stochastic sequences, are examined.

4.2 PROBLEMS AND AIMS OF TIME SERIES ANALYSIS

In terms of using the previous described mathematical models, it is possible to distinguish the two main classes of problems and methods for handling and analysing financial time series [5, 7, 8]: the identification and forecasting problems. Identification problems arise when trying to find the parameters of the system being considered, that generates a particular time series. These parameters can vary from statistic distributions in the models being considered to spectral and fractional characteristics of the system. These parameters could help to classify the system, and/or distinguish standard behaviour from anomalous – crises, disasters, catastrophes, etc. The problem of finding (measuring) some characteristics of a system can be also be used within the identification problem. In particular, these problems embrace attractor dimension, Lyapunov parameters, entropy, etc. Sometimes, when solving the identification problem, the tasks of examining the topological attractor's property, and/or motion (evolution) equation of approximation on experimental data has to be worked out to understand the mechanisms of the bifurcations, and to find correct approximation models. The identification problem arises within some system classification tasks (e.g. for diagnostics purposes). Here, it is usually unimportant whether to look for invariant characteristics of the system, or to deal with some other parameters, suitable for limited set of problems only. It is only important to be able to classify effectively and quickly. Another set of identification problems arises. These include determining the type of data, the researcher is dealing with. Computing algorithm, which works without distinguishing data types, can provide necessary estimates both in terms of determined systems, and in terms of random processes, or any other. So one can make drastic mistakes when one algorithm is used instead of another – the correct one, and thus giving wrong results applicable for some classification purposes. Forecasting problems can arise when the results of the previous observations are used to predict (with some certainty) future values of observable characteristics, or, even broadly, the future status of the considered system. This problem in particular includes tasks of dynamic forecasting of time series, which in fact turns into the problem of a dynamic system reconstruction by a time series. Here, motion equations are not of a great interest, and are usually considered as black boxes. The more important thing is that the output signal has to follow the input signal without errors. Such problems are typical for approximational neural networks after the network study is over. Taking into account the nature of this research, it is worth focusing here on the identification problems only, leaving the problems of forecasting for further research.

4.3 Statistical Methods of Time Series Analysis

Time series have been a focus of statistical research for a long time, so the methods of their handling are widely known [4, 5, 8, 9, 10, 14, 18, 26, 385, 386]. In the case of price characteristics, significant difficulty can arise since it is not known exactly which mathematical model should be associated with the results. Due to this uncertainty, time series have been often analysed using methods of mathematical statistics. The terms *average*, *absolute ratio*, *relative ratio* of the time series have became widely recognised and used together with the terms *distribution*, *statistical estimate* of their parameters, etc. Many of these statistical methods have become traditional. Also, the mathematical techniques that embraces the concepts of a *random sequence* and *processes*, *statistical differential equations*, etc. is still in active use. All these concepts and methods have been generalised for financial mathematics and statistics [4, 5].

The idea of a statistical model is the core idea within most methods of time series data analysis. It is systems, which have "standard", mostly uncorrelated Gaussian noise, and the output data has its own distribution and own time correlations, resulting from dynamics. The research problem lies in developing of dynamic part of the model in such a way, that noise is transformed into a time series, similar to the considered one. With regards to this research, statistical characteristics, methods and models can be considered. For example, Table 4.1 presents descriptive statistics for distribution of $h_i(\Delta)$, found for different discretisations Δ in the period between 01.01.1987 and 31.12.1993, when 8,238,532 ticks have been recorded for DEM/USD exchange rate [6].

Table 4.1. Descriptive Statistics for $h_i(\Delta)$

Δ	N	$\bar{h}_N$	$\hat{m}_2$	$\hat{S}_N$	$\hat{K}_N$
10 minutes	368,000	-2.73*10 ^{7}	2.62*10 ^{7}	0.17	35.11
1 hour	61,200	-1.63*10 ^{6}	1.45*10 ^{6}	0.26	23.55
6 hours	10,200	-9.84*10 ^{6}	9.20*10 ^{6}	0.24	9.44
24 hours	2,100	-4.00*10 ^{5}	3.81*10 ^{5}	0.08	3.33

Here N is the number of points of the sample space of type $t_i = i\Delta$; $\bar{h}_N = \frac{1}{N}\sum_{i=1}^{N} h_{t_i}$; $\hat{m}_2$ is the empirical variance; $\hat{m}_k = \frac{1}{N}\sum_{i=1}^{N}(h_{t_i} - \bar{h}_N)^k$ is the k^{th} empirical momentum; $\hat{S}_N = \hat{m}_3 / \hat{m}_2^{3/2}$ is the empirical skewness; and $\hat{K}_N = \frac{\hat{m}_4}{(\hat{m}_2)^2} - 3$ is the empirical kurtosis. Positive skewness indicates a distribution with an asymmetric tail extending towards more positive values. Absolute mean values are far less than the standard deviation, and thus can be considered as equal to zero. So it is not likely to support the hypothesised normal distribution of characteristics, since kurtosis is too high, and grows with declining discretisation Δ. As kurtosis is calculated as the fourth momentum, this allows us to assume that the distributions have "heavy" tails. It is possible to see that when Δ increases, the number of components determining h_i also increases, which raises the likelihood of a Gaussian distribution of h_i.

The method of Ordinary Least Squares (OLS) is essential in a regression analysis [4, 5]. This method minimises (equals positive and negative values) absolute errors [126], although the relative (ratio) errors at the beginning and at the end of the considered range of variables may vary significantly. The lack of control over these errors can lead to substantial inaccuracies. Being a non-robust procedure, OLS is very sensitive to statistics' heterogeneity, i.e. even for a small number of drastic anomalies in the statistical data [126, 127]. On the one hand, this requires implementation of front-end handling methods, aimed at finding and barring "suspicious" values in the statistical data. On the other hand, this allows us to use OLS for the identification and classification of those moments, at which anomalies appear/disappear. For illustration purposes of OLS properties, let us look at the (simplest) linear regression. Usually, two types of relations are distinguished – functional and stochastic. If all experimental data belong to the direct regression line $y = a_0 x + a_1$, there is a functional relation. In a functional relation correlation between y and x makes no sense, since the pairwise correlation coefficient ρ always equals unity. When there is noise or there are some random errors in the system, correlation between y and x is stochastic. Now, finding a pairwise correlation coefficient between y and x, and a statistical estimation of this coefficient, are very important procedures, which provide useful information about the considered relationship. The pairwise correlation coefficient ρ may vary from (−1) to (+1). When ρ approaching (+1), the functional relation in the considered sequence increase. If the variables are random and normally distributed, then there are two regressions. The first determines $\hat{y}$ from x, and the second $\hat{x}$ from y. The lines of these two regressions intersect in the centroid of points $(\tilde{x};\tilde{y})$, and generate "scissors". The less open are these "scissors", the more close are the stochastic relationships to functional. This means that results obtained via the OLS estimation are irreversible, and there are two regression models. The first model $\hat{y} = a_{0(yx)}x + a_{1(yx)}$ is called the direct regression, and the second model $\hat{x} = \dfrac{y}{a_{0(xy)}} + a_{1(xy)}$ is called the inverse regression.

The equation $\hat{y} = a_{0(yx)}x + a_{1(yx)}$ is not algebraic, allowing x to be found, because this model was obtained through the minimisation of the sum of squared deviations along the Y axis. To find x, the sum of the squared deviations in the inverse regression has to be minimised along the X axis. When analysing results of linear combinations, experimental data are usually organised in the form of an ellipse of dispersion. Although lines of direct and inverse regressions intersect the spin axis of the ellipse $y = k_0 x + b$ in its centroid of points $(\tilde{x};\tilde{y})$, they never coincide with the spin axis. The line of direct regression comes closer (flat) to the X axis than the spin axis of the ellipse of dispersion, and consequently the inverse line is steeper along the X axis. When $a_{0(yx)} < k_0 < a_{0(xy)}$ and $a_{0(yx)} \neq a_{0(xy)}$, the composition $a_{0(yx)}\left(\dfrac{1}{a_{0(xy)}}\right) \neq 1$, but rather equals $\dfrac{a_{0(yx)}}{a_{0(xy)}} = \rho^2$, where ρ is the *coefficient of cross-correlation* between x and y (*pairwise correlation coefficient*). By the same reason, the slope coefficient of the spin axis of the ellipse of dispersion can be presented in the form $k_0 = \dfrac{a_{0(yx)}}{\rho}$ or $k_0 = a_{0(xy)}\rho$. When ρ is close to unity, i.e. when the dispersion of the experimental data is small and the ellipse of dispersion is oblong, both regression lines are close to the spin axis of the ellipse, and the difference between these two lines can be neglected. When the coefficient of correlation is small (following [126], $\rho \leq 0{,}96$), this difference increases dramatically, and the OLS method may become ineffective.

In this case it is more efficient to apply other methods [126], e.g. the method of orthogonal regression, which uses mean $\overline{\rho}$, $\bar{x}$ and $\bar{y}$, and is modelled through the following equation [127]:

$$\hat{Y} = \bar{y} + \frac{2\overline{\rho}}{\sigma_0 + \sqrt{\sigma_0^2 + 4\overline{\rho}^2}}(X - \bar{x}), \tag{4.1}$$

where $\sigma_0 = \frac{\sigma_x}{\sigma_y} - \frac{\sigma_y}{\sigma_x}$, and $\overline{\rho} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}}$. The main idea of the method of orthogonal regression lies in measuring the distance from the experimental points to the line of orthogonal regression with a perpendicular between these points and the line. The idea of orthogonal regression, like the ideas of direct and inverse regressions, is tied-up with the ellipse of dispersion. The line of orthogonal regression (4.1) represents one of the main spin axes of the dispersion ellipse, and two other regression lines represent diameters of this ellipse. On the one hand, these representation may have beneficial effects for symmetrical fields of dispersion (in relation to the desired regression line), but, on the other hand, for asymmetrical fields these effects become negative because of systematic errors (whose size is not always possible to find), overstate or understate the desired estimate. Due to this, and taking into account some other factors [126, 127] like a lack of scale invariance of this method, the application area of the method of orthogonal regression has to be considered as having a number of restrictions. Since the choice of the regression equation is optional, and depends on which random variable – x and/or y – to consider initially as fixed, it is possible to use any of these regression equations. The choice of a particular regression equation has to be thoroughly justified. Here, equations of direct and inverse regression can be considered as determining correspondingly the lower and the upper boundaries of the desired regression equation. Orthogonal regression can only be considered as a rough approximation to the regression equation.

On the FX-market at each point of time t, two new quotations – the *ask price* (S_t^a) and the *bid price* (S_t^b) are specified. The difference ($S_t^a - S_t^b$) is called the *spread*. The spread is positively correlated with the price volatility (considered here as a standard deviation) [6]. Growing volatility, which increases the risk with a less accurate forecast for the price variation, raises the spread in compensation for the high risk. If prices S_t^a and S_t^b are introduced in the form

$S_t^a = S_0^a \exp(H_t^a)$ and $S_t^b = S_0^b \exp(H_t^b)$, and assume that $H_t = \dfrac{(H_t^a + H_t^b)}{2}$, $S_t = S_0 \exp(H_t)$, where $S_0 = \sqrt{S_0^a S_0^b}$, it is possible to get logarithms of the geometrical mean $H_t = \ln\sqrt{S_t^a S_t^b}$ and $S_t = \sqrt{S_t^a S_t^b}$. These parameters are exactly the prices $S = (S_t)$ and their logarithms $H = (H_t)$, which are used for the analysis of exchange rates, after combining two real prices S_t^a and S_t^b into one S_t. The stochasticity of the variable S_t is determined with both the nature of components included into S_t, and with the presence of some additional random error. This error emerges for example because real bid prices are usually confidential and are not disclosed. Thus, price characteristics can not be unambiguously considered as determining parameters (independent variables) for time series research.

Similar reasoning can be presented for the time of price-fixing. Price characteristics, fixed in daily charts, are usually adjusted to the time when the trade session ends. However, this does not necessarily imply that just this price was announced at the same time. In other words, the operational time of a trade session also could not be considered as independent variable for time series analysis. If we consider the (existing) difference between operational and astronomical time, this last conclusion becomes even more evident. The problems of applying statistical methods for analysing various characteristics, and defining sound stochastic relationships and the optimal choice of regression equations has not been fully examined in the financial literature. So, the problem of the optimal variable selection, including finding the correlation between them, will be addressed as the main research interest of this applied study.

The other important statistical research method, applied in financial time series analysis, is statistical *R/S Analysis*. This method is robust and allows us to find properties in statistical data such as: clustering, persistence, strong after-effect, long memory, quick antipersistence, self-similarity (fractionality), existence of periodic and non-periodic cycles, ability to distinguish stochastic and chaotic nature of noise, etc. In accordance with this, besides the work [100], it is worth emphasising the works of Mandelbrot and his co-authors [83, 119, 128,..., 138], who did an outstanding piece of work developing *R/S Analysis*, its methodology and application fields. Two studies of Peters [58, 59] have to be mentioned as containing material on the application of *R/S analysis* to some sectors of the financial market. Let us consider only the main characteristics of practical application of *R/S Analysis*. If it is assumed that the analysed data $h_1, h_2, \dots, h_n$ ($\mathrm{E}h_i = 0$, $Dh_i^2 = 1$) are IID, then the distribution $Q_n = R_n / S_n$ is independent from the mean and the variance of h_k, $k \le n$. This nonparametric condition leads to the criterion, allowing (at some level of significance) for the rejection of the core hypothesis (H_0) that considers variables follow a random walk. The basic idea of this criterion, based on *R/S* statistics, is the following [59, 100, 138]. If H_0 is true, then for high *n*, the ratio R_n / S_n has to be close to $\mathrm{E}_0(R_n / S_n) \sim \sqrt{\frac{\pi}{2}n}$. From this ratio $\ln(R_n / S_n) \approx \ln\sqrt{\frac{\pi}{2}} + \frac{1}{2}\ln n$. Thus, if H_0 is accepted, values of $\ln(R_n / S_n)$ have to group along the line $\ln\sqrt{\frac{\pi}{2}} + \frac{1}{2}\ln n$, based on a logarithmic system of coordinates $(\ln n; \ln(R_n / S_n))$. If presenting the available statistical data (points $\ln n$, $\ln(R_n / S_n)$) on the corresponding logarithmic scale, and, using OLS estimation, to draw the line $\bar{a}_n + \bar{b}_n \ln n$ through them, then hypothesis H_0 can be rejected if $\bar{b}_n$ is significantly different from ½. It is worth emphasising, that it is not easy to find the level of significance of the difference of $\bar{b}_n$ from ½, since it is difficult to obtain a R_n / S_n distribution for finite *n*. If experimental results support the condition $R_n / S_n \sim cn^H$, for $H \neq ½$, it is worth questioning for which models this condition holds. For instance, for $0 < H < 1$, $H \neq ½$, it could hold for models, considered as fractional Gaussian noise. On the contrary, when $H = ½$, we have classical Gaussian noise.

When doing *R/S Analysis*, it is useful to introduce $Q_n / \sqrt{n}$ statistics. This type of analysis is based on the fact, that in the case of noise with $H = ½$ (i.e. for brown noise [38]), statistics $Q_n / \sqrt{n}$ have to be stable for high values of *n*. If there is fractional Gaussian noise with $H > ½$ (i.e. black noise [38]), statistics $Q_n / \sqrt{n}$ increases with larger *n*; and, on the contrary, if $H < ½$, $Q_n / \sqrt{n}$ declines. *R/S Analysis* allows us to reveal periodic and non-periodic cycles in the considered time series [58, 59, 130, 132]. For periodic and non-periodic cycles the spread during the following cycles can not increase significantly in comparison with the size of spread during the first cycle. Similarly, empirical variance also stabilises with cycles. As a result, when new cycles occur, the behaviour of $\ln(R_n / S_n)$ changes dramatically – R_n / S_n statistics stabilises. This fact explains why *R/S Analysis* can be applied for revealing cyclic effects. Thus, *R/S Analysis* has to be considered as one of the main research techniques, and results obtained can be a good basis for comparison with results from other methods.

4.4 TIME SERIES ANALYSIS AND HANDLING USING METHODS OF NON-LINEAR DYNAMICS

It becomes clear that it is not possible to apply just one "universal" approach or method of non-linear dynamics for the analysis of time series simply because there is no such "universal" technique [7]. At the same time, in non-linear dynamics there are some core ideas, methods and algorithms, that allow us not only to find some general or peculiar characteristics of the considered system, but also to determine their precise quantitative characteristics. It is firstly worth mentioning the methods of visual presentation of experimental data, widely applied in non-linear dynamics [7, 10, 11, 12, 16, 17, 19, 23]. Phase diagrams are widely used for the presentation of results. Another field of research is focused on the estimation of the dimension of a dynamic system, particularly, the attractor's dimension [7, 37, 38, 39]. The dimension indicates the complexity of the dynamic system and its attractor, i.e. it answers the fundamental question, what is the minimal number of variables that have to be included into the model, or, at least, provides an estimate of this number (usually lower estimate). Dimension is also one of the characteristics which can be applied to both Euclidian space and its manifolds (tori, spheres, etc.) and fractional sets. Dimensions are also important because they are among those rare characteristics that sometimes can be obtained for real time series, thus providing researchers with valuable experimental parameters of dynamic system. There are many types of dimension. For instance, it is possible to distinguish probabilistic dimensions [7, 38, 39], spectral dimensions [7, 38], dynamic dimensions [37], etc. However, for this research, not all of them are of the same importance, and we will consider some of the most frequently used ones [7, 37, 38, 39].

Let us now turn to geometrical dimension, where one of the most important characteristics is the *topological dimension* d_T, which is equal to n for an n-dimensional Euclidian space R^n, or for an n-dimensional manifold, locally equivalent to R^n. This dimension can be found for more complicated sets like a Cantor set ($d_T = 0$) [7]. The first general definitions of dimension for compact and metric sets were based on the definition of *induction*, so, sometimes, dimensions were also referred to as *induction dimension* [37]. Thus, the empty set has a dimension of (-1), a point (or ensemble of points) has dimension 0, the line has dimension $(+1)$, etc.

Another approach to the concept of dimension is based on set cover. If there is compact X, its cover with open sets of size no more than ε, then the dimension is the smallest number n such that for any $\varepsilon > 0$, there is a cover, where any point from X belongs to no more that $n+1$ covering sets. It becomes clear why the topological dimension of a Cantor set equals 0 – because this set can be covered with nonintersecting segments in such a way that each point belongs to only one segment.

An important type of dimension, which allows for the definition of a fractional set, is the *Hausdorff dimension* d_H. This dimension also relates to the cover of the set or metric space with sets A_i, with $\mathrm{diam}\, \mathrm{A_i} < \varepsilon$. Let $m(\varepsilon, p) = \inf\limits_{\{A_i\}} \sum\limits_i (\mathrm{diam}\, \mathrm{A_i})^p$, where the infimum is taken over all possible covers with $\mathrm{diam}\, \mathrm{A_i} < \varepsilon$. Then, the Hausdorff dimension is $d_H = \sup\{p : \sup\limits_{\varepsilon>0} m(\varepsilon, p) > 0\}$. For the Cantor set: the topological dimension $d_T = 0$, and the Hausdorff dimension $d_H = \log_3 2$, where $d_T < d_H$. The sets for which $d_T < d_H$, are called *fractional sets* or *fractals* [7]. If the set belongs to an *n*-dimensional Euclidian space or manifold, that the diameter of the cover set is calculated from metrics of this set or manifold. The value of d_H may depend on the way the diameter is calculated, i.e. on the metrics used, and there are different ways of obtaining these metrics, therefore d_H can vary. Thus, the definition of fractal requires fixed metrics. d_H has a rather theoretical than a practical application because in most cases it is impossible to calculate it as its definition does not allow for any quantitative algorithms (e.g. due to the inability to calculate an exact infimum for all covers). As a result, quantitative estimates are usually made on other parameter – the *limiting capacity* d_C [7], often also referred to as the *Minkovsky dimension* – d_M [37], or the *Minkovsky-Buligan dimension* [38, 39].

Consider there is a set A, and also there is a metric which measures of distances. Let us also define $N(\varepsilon)$ as the minimal number of spheres with diameter ε, necessary to cover A. Then, the limit $d_c = \lim\limits_{\varepsilon\to 0} \dfrac{\log N(\varepsilon)}{\log(1/\varepsilon)}$ is called a *capacity of the set A* (if this limit exists). If this limit does not exist, then the lower and upper boundaries of the set capacity are introduced as the infimum and supremum. For a Cantor set $d_C = d_M = d_H = \log_3 2$. However, for any random set it is only possible to state that $d_C = d_M \geq d_H$, as (for example) for the ensemble of points of the sequence $x_n = 1/n$, for which $d_H = 0$, but $d_C = d_M = ½$. The dimension d_C is sometimes referred to as the *metrics dimension* [7]; the *fractal dimension* or the *dimension of the realisation process* in relation to motion processes; and the *fractional dimension* or the *self-similarity di-*

mension in relation to self-similarity fractals [37], etc. The definition of the capacity of d_C is used in most quantitative methods of dimension estimation [37]. When finding $N(\varepsilon)$ for different values of ε, and taking into account the definition of capacity, where $N(\varepsilon)$ has to follow $\sim \varepsilon^{-d_C}$ for small values of ε, and consequently $\log N(\varepsilon) \cong -d_C \log \varepsilon$. Thus, the capacity estimation turns into the search for the "most linear" part of $\log N(\varepsilon)$ as a dependent variable from $\log \varepsilon$, and to the construction of a linear approximation of the type $\log N(\varepsilon) \cong b \log \varepsilon + c$ on the considered segment with an OLS estimation techniques. After that, $d_C = -b$ can be taken as the capacity estimate. It is evident that in further research estimation, techniques for the dimension d_C, regardless of its calculation and application properties, can be used. Unless there are serious objections, the relatively simpler notion of this dimension – d will be used.

Let us also consider another conceptual field of research in the field of non-linear dynamics, associated with the idea of the reconstruction of the attractor from experimental data. The idea of using autoregression analysis has been employed for quite a long time, but, possibly, the link between autoregression models and dynamic systems has not been fully studied. This issue was raised only by the end of the 1970s during experiments aimed to support the ideas of non-linear dynamics with small modes. In [139] the authors showed how to obtain a fair geometric configuration of a strange attractor with a small dimension if instead of variables x, included into the equation of the system, m-dimensional vectors are used, formed, from time series components, in the same way they are formed in autoregression problems:

$$Z_i = \{x_i, x_{i+1}, \ldots, x_{i+m-1}\} \tag{4.2}$$

One year later, Takens [140] formulated a theorem, based on the grounds of the recognised Whitney theorem from differential geometry [141]. Since Takens theorem underlies nearly all contemporary algorithms of time series analysis with methods of non-linear dynamics, we will examine it in more detail (we will introduce only the most important terms here).

A K-dimensional manifold is a generalisation of the notion "smooth k-dimensional surface in n-dimensional space, which can be locally parameterised with k Euclidian coordinates in a neighbourhood of points". A K-dimensional manifold could be a sphere, a torus, etc. If the same manifold is considered, but without the n-dimensional space – this will present a manifold as an abstract mathematical object (M^k). When this manifold is implemented in an n-dimensional space as a surface S^k, which does not intersect with itself, then it is said to be embedded into R^n. This embedding could be presented by means of a differential vector function

F in M^k, for which the representation $M^k \to S^k$ is unambiguous (biunique). There is also an inverse differential function F^{-1}, representing S^k backwards into M^k, i.e. $S^k = F(M^k)$ and $M^k = F^{-1}(S^k)$. Here F^{-1} is defined only in S^k, otherwise it could not be an unambiguous function. By choosing different F and n, it is possible to get various representations of the manifold.

Let a vector function be defined in the manifold M^k or some surface S^k. This vector function is differentiable as many times as needed, and represents M^k into an m-dimensional Euclidian space R^m. What exactly will be in R^m? Let M^k be a manifold that is at least twice differentiable, $g(x)$ be some twice differentiable function, representing $M^k \to R^m$ for which the matrix of derivatives $\frac{\partial g_i}{\partial x_j}$ has rank k. The last condition is necessary to get an object, which after representation, has size no less than its initial size (e.g. to prevent representation of a surface into a line). Then, this representation will immerse the manifold M^k into R^m if $m \geq 2k+1$ (Whitney theorem, [115]). Immersion is locally identical to embedding, but can have self-intersections, and thus it is overall impossible to find an inverse representation. For instance, if a circle was considered as a manifold, then an ellipse on the surface will be an embedding, and a figure-of-eight will only be an immersion. This occurs because two different points of this circle correspond to one point on the figure-of-eight. That is why Whitney's theorem is insufficient to justify these methods of time series analysis.

The only thing left to be clarified is which manifold M and which function $M \to R^m$ are to be considered, and how this relates to time series. Let $\varphi^t(x)$ be a dynamic system with phase space P. Let values, combined into time series, be parameters of some "observed" scalar function of the state of the system $x(t)$: $x_i = h(x(t_i)) = h(\varphi^t(x_0))$. Whitney's manifold M can be presented in the form of a phase space P, or as any invariant manifold M^d from P. In most cases, when all transition processes end and it is possible to consider the trajectory as being on the attractor, it is convenient, for example, to consider the so-called minimal inertial manifold (MIM) as M^d [7]. The ideas of an inertial manifold, justifying finite dimensionality of attractors in dissipative dynamic systems, are closely related to the concepts of parameter degree and the principle of subordinacy of modes in synergy. These concepts give a possibility for finding a solution of the equation in partial derivatives as $t \to \infty$ via some finite dynamic system. The term of an inertial manifold [142], possibly, originates from hydrodynamics, and the term manifold emerges because from a geometrical point of view, trajectories, without occupying all feasible phase space, are attracted to some surface of finite (sometimes small) dimension – manifold. This asymptotically stable (i.e. with exponentially attractive trajectories) positively invariant differential manifold, containing the system's attractor, is now called the *inertial manifold*. The existence of a system's inertial manifold allows the analysing of asymptotic behaviour not of the system itself, but of its inertial form to be analysed, i.e. of the dynamic system of smaller dimension. For that, the following things have to be remembered. The inertial manifold is not-unique. Usually, there is a sequence of self-embedded inertial manifolds and corresponding inertial forms. Among them there is a single manifold of minimal dimension, which, following from [7], will be referred to as a *minimal inertial manifold*.

MIM can be characterised with some dimension d_N. Therefore, if a system has MIM with dimension d_N, its asymptotic behaviour is described with d_N variables. This is one of the most important characteristics of a dynamic system and of the "complexity" of its behaviour. All estimations of fractional dimensions for attractors of systems usually provide lower estimates of d_N. It is usually referred to as the *geometrical dimension* [143], the *dynamic dimension* [144], the *number of order parameters* [145], the *local internal dimension* [146], or the *number of independent degrees of freedom* [147]. In the latter case, this is often $\frac{d_N}{2}$, because this term originates from mechanics, where there are two variables for one degree of freedom. It is evident that d_N is interesting as a classification parameter for the systems, but there are no appropriate methods for such analysis and classification can actually be too rough for most practical cases. In the theory of finite-dimensional dynamic systems the notion of inertial manifold is barely used, because the notion *attractor* is more significant. Necessity for using it comes only when analysing time series, and when considering methods of attractors' reconstruction.

Let us now examine the construction of z-vectors from (4.2) in more detail. When the time step between elements of a time series equals τ, and the vector $x(t_i)$ for simplicity is defined as x_i, then $x_{i+1} = \varphi^{\tau}(x_i), x_{i+2} = \varphi^{2\tau}(x_i), \ldots, x_{i+m-1} = \varphi^{(m-1)\tau}(x_i)$. Therefore:

$$
\begin{aligned}
x_i &= h(x_i) \equiv \Phi_0(x_i), \\
x_{i+1} &= h(x_{i+1}) = h(\varphi^{\tau}(x_i)) \equiv \Phi_1(x_i), \\
&\ldots\ldots\ldots\ldots \\
x_{i+m} &= h(x_{i+m}) = h(\varphi^{m\tau}(x_i)) \equiv \Phi_m(x_i).
\end{aligned}
$$

We have linked all the components of the vector z_i with one and the same state of the dynamic system x_i. Therefore, there is some vector function (following Takens, we define it as Λ), which maps the vectors $x_i \in M^d$ into points of the m-dimensional Euclidian space R^m, $z_i = \Lambda(x_i)$, $x_i \in M^d$, $z_i \in R^m$. We have approached the situation, described in Whitney's theorem, where the representation g is played by Λ, and the manifold is represented by M^d. Whitney's theorem hypothesises that M^d, h and φ^τ are at least twice differentiable, and that for all fixed points and cycles with period $k\tau$, $k < d$, all eigen-values are prime numbers not equal to unity, and the $h(x)$ for them are different. Then, Takens theorem states that typically, i.e. for a general representation of Λ for $m \geq 2k+1$, Λ will give an embedding of M^d into R^m. If M^d into R^m is defined as $S^d : S^d = \Lambda(M^d)$, then, following the theorem, this generally does not have self-intersections. Following [7], the embedding implies that:

- The function Λ is differentiable and has an inverse differentiable function Λ^{-1}, defined on S^d: $M^d = \Lambda^{-1}(S^d)$;
- Each trajectory of the dynamic system has a corresponding image in Z-space. These images have the same properties as the initial trajectories, so only one Z- trajectory will go through each point S^d;
- A dynamic system can be defined on S^d. Indeed, $x_i = \Lambda^{-1}(z_i)$, $x_{i+1} = \varphi(x_i)$, $z_{i+1} = \Lambda^{-1}(x_{i+1}) = \Lambda(\varphi^\tau(\Lambda^{-1}(z_i)) \equiv \Psi(z_i)$, $z_i \in S^d$, where ψ represents S^d into S^d, and outside S^d surface, the mapping ψ is not defined. If the last component of this expression only is left, we get another case, which can be described as "the representation with delay" or "a non-linear autoregression": $x_i = F(x_{i-1},...,x_{i-m})$. This expression can be used for time series analysis and forecasting;
- Therefore, we have two mappings $x_{i+1} = \varphi(x_i) \equiv \Phi(x_i)$, $x_i \in M^d$, $\Phi : M^d \to M^d$ and $z_{i+1} = \Psi(z_i)$, $z_i \in S^d$, $\Psi : S^d \to S^d$, which can be considered as mappings, connected with a non-degenerated and invertible change of variables $z = \Lambda(x)$, or as different representations of the same mapping. Thus, characteristics, invariant to non-degenerated change, have to coincide for both systems. In particular, among these characteristics are fractional dimensions of the attractor. Therefore, it is worth trying to find the described characteristics within experimental data without knowing all the variables of the dynamic system. Also, it is possible to approximate the function $\Psi(z_i)$.

A "typical nature" has to be considered as, for example, the statement: "generally two lines intersect". A similar notion can be made to Takens theorem. This theorem is based on the assumption that in the general case, the rank Λ equals d. In the "typical" case there should be no self-intersections. However, proving each time that there are no self-intersections is not a trivial task. Therefore, while analysing experimental data it can be hypothesised that there are no self-intersections. If there are a small number of self-intersections, their presence does not affect the results. For instance, for a circle and a figure-of-eight the estimates of dimension will be the same. Practically, this implies that proper immersion (with a minimal number of intersections) could be as good as embedding. However, a sufficient number of self-intersections can radically change the results. Furthermore, there could be such delays in τ of the variable $h(x)$ of the dynamic system $\varphi'(x)$, that there will be no embedding. Although this situation is not typical, the chance of facing it is still quite high. For instance, when the phase space is represented as a point (rank Λ less than d), then τ is equal to the period of the process.

It does not follow from Takens theorem that the vectors in equation (4.2) always allow the properties of dynamic systems to be analysed. This is (usually) possible, but there are a lot of exceptions. The main research object while analysing time series is not the dynamic system itself, but its minimal inertial forms on MIM. It is sufficient to say that Takens theorem provides strong mathematical grounds for the concept of non-linear autoregression. Some (similar) statistical concepts are referred to as *reconstruction of attractor*; *reconstruction of the phase space*; *reconstruction of the dynamic system*; *forecasting with methods of non-linear dynamics*; etc. [7]. It is possible to say that non-linear dynamics provided a number of new approaches to analysing time series and new characteristics of systems, which can be used for identification purposes and will be used in this research.

The practical application of reconstruction is often accompanied by numerous problems. These problems emerge because intervals of the time series are limited. Firstly, problems arise due to limitation of keeping of information. Secondly, the speed of data handling is not fast enough. Thirdly, stationarity of the objects being considered is a concern, because it is always important to know for how long it is possible to assume that the same system is considered, since after $\varphi'(x)$ changes, the vectors z may be constructed differently. Thereupon, it is worth noting that the problem of stationarity arises for most dynamic methods of time series analysis. Of high importance is the problem caused by the discontinuity of the time series. For simplicity we can assume that there is a time series of size N, which are values of some observed parameter. Then, the reconstructed z-vectors (4.2) give $N-m$ points on the surface $S^d \in R^m$, carrying information on the dynamic system $\Psi(z)$ and its attractor. Here, the amount of information obtained depends on the properties of this surface (warping, twisting, etc.) and on the properties of the function $\Psi(z)$ (magnitude of derivates, etc.). Once we have a finite number of points, there is some distinctive distance l between any point and its neighbours (environs). Thus, smaller scales for this time series are indistinguishable. A similar problem, but in a slightly different form, could be found in digital signal processing [90], where when the time interval between readings is considered equal to Δt, frequencies greater than $\frac{1}{2\Delta t}$ are not representable.

However, in reconstruction problems the properties of S^d and $\Psi(z)$ are *a priori* unknown, therefore similar estimations when curvature or derivatives are less then $\sim 1/l$ are not feasible. Therefore, the problem of optimal choice of reconstruction parameters has to be formulated to obtain the most informative set of reconstructed vectors. Some quality criteria for the estimation of reconstruction descriptiveness are required. Generally, the properties of S^d and $\Psi(z)$ are dependent on the dynamic system φ, the observable variable h, and the delay τ. Also, the dimension of vectors m has to be added. (Sometimes, this dimension is referred to as the *embedding dimension* [7].) Usually, the first two factors can not be changed, and the time step is also fixed for the representation, although, τ generally could be changed. Regarding the dimension of the embedding m, Takens theorem requires this dimension not to be small, but does not impose any limitations on the upper boundary. This implies that *de facto* we can operate with only two variables m and τ. Previous research [148, 149, 150] contains numerous recommendations and approaches to the choice of delay τ and embedding dimension m. However, all these recommendations are not too reliable because for simple simulations they do work, but for experimental time series results are often misleading. Nonetheless, following [7], we will try to formulate some general recommendations and principles of finding m and τ.

If the reconstruction of the trajectory of the system is focussed upon, then only one trajectory has to pass through each point, i.e. perfect reconstruction does not have to contain self-intersections. It is clear that self-intersections in a manifold of discreet points z_i are unlikely, therefore we have to look at "false neighbours" – the pairs of vectors that are close in the reconstruction while their prototypes are actually far from each other. Two procedures are considered for finding these pairs, though actually they reduce to a single method. Let $Z_i^{(m)}$ and $Z_j^{(m)}$ be two neighbours in the reconstruction of dimension m, and $Z_i^{(m+1)}$ and $Z_j^{(m+1)}$ be the corresponding values in reconstructions $m+1$. If we are dealing with two really close neighbours, they (with rare exceptions) are close in both reconstructions. At the same time, false neighbours in reconstruction m, usually, fall apart with increasing m. The pairs, for which $\left\|Z_i^{(m)} - Z_j^{(m)}\right\|$ is small, and $\left\|Z_i^{(m+1)} - Z_j^{(m+1)}\right\|$ is not, are called *false nearest neighbours* (FNN). If now to maximise and estimate the number of FNN, it can be seen that its number falls down sharply when the required dimension for the correct reconstruction is achieved. Consequently, this minimisation allows the minimal value of m to be found. Nearly the same method can be obtained as a result of a different discussion. If the reconstruction for given m is correct, the representation $\Psi(z)$, $z_{i+1} = \Psi(z_i)$ has to be continuous and differentiable. Then for close vectors z_i and z_j their representations z_{i+1} and z_{j+1} have to be close too. Thus, we get slightly a different definition of FNN, although *per se* we talk about the same elements of the time series. The second method clarifies where we get a small number of FNN in the correct reconstruction – these are simply those places, where the derivative $\Psi(z)$ is large, or (what is the same) where trajectories fall apart quickly. However, this method does not allow for any practically useful formal statements, since it is based on informal notions of "close" and "non-close" pairs. It is impossible to formally define these pairs, as the properties of $\Psi(z)$ are *a priori* unknown. Therefore, for numerical implementation of this method it is worth relying on a phenomenological approach, based on heuristic considerations. Sometimes the lack of false neighbours in a reconstruction is indicated indirectly, e.g. while measuring some parameter for different m and looking for when this parameter becomes independent of m.

Initially, the way of choosing τ was based on the idea that if the components, which determine the vector, are independent of each other, then the reconstructed vectors carry maximum information on the system. The simplest way of getting such linear independence is in choosing τ close to the first zero of the autocorrelation function for the sequence x_i. But in this case uncorrelatedness does not necessarily imply independence. The method, founded on information theory, which uses the first minimum of the reciprocal information for $x(t)$ and $x(t+\tau)$ is admittedly more complicated [149, 151, 152]. For a simple model system nearly all reasonable choices of τ are not bad if not good. For more complex systems these methods are often totally inapplicable because functions, used in them, are monotonely decreasing with increasing τ, and do not have the first zero or minimum. Sometimes a zero or minimum exists, but the choice of τ is not the optimal one. Therefore, a number of approaches was taken to develop more advanced methods, e.g. similar to the technique of choosing the dimension of embedding m. As most of the attempts resulted in ambiguous results [7], we focus on the most important of them.

The most interesting thing is that the quality of the reconstruction is affected not by τ itself, but by the time series, embraced with vector z between its first and last elements. Following [7], we refer to it as the *reconstruction window*, and define it as $w = (m-1)\tau$, assuming that m is sufficiently large to match the criteria of Takens theorem. The influence of w on the reconstruction is somewhat different from the influence of m, though it could also be sometimes interpreted in terms of false neighbours. Nonetheless, it is more appropriate to use the term *reconstruction distortion*. It is possible to distinguish two types of distortions. The first type emerges when w is very small, and shows up in systems with continuous time. According to this type of distortion, the image of the reconstructed set is "compacted" along some direction. This, firstly, produces false neighbours. Secondly, very small scales have to be distinguished for studying the attractors, and subsequently requires large samples, and the time of change has to be of the same order as the time of returning to the very small neighbourhood of a point. The second type of distortion emerges when w is too large, but only for chaotic systems, where time discreteness or continuity is not important. This type of distortion is somewhat similar to "Smale's horseshoe", when the set is strained and folded. For some scales, instead of analysing the structure of the set, the structure of the folds is analysed. At the same time, false neighbours could emerge on neighbouring folds, and, as a result, on large scales the reconstruction may look a much larger object than it actually is. At present there are no exhaustive methods for making the optimal choice of reconstruction parameters, and there is no universal criteria for testing the quality of the reconstruction. Any known method or criterion for choosing m, τ or w is not a dogma, but only a guide to action. The objective of this work for choosing the optimal observable parameter for the reconstruction of a currency exchange rate time series is justified.

In conclusion let us turn to the consideration of the forecasting problem, which, probably, is one of the oldest problems of time series analysis [4,..., 19, 22,..., 26, 29, 30, 31, 36,..., 39, 48, 49, 50, 58,..., 62, 85,..., 89, 91,..., 94, 153, 154], and let highlight some peculiarities of approaches, based on non-linear dynamics only. Suppose there is a reconstruction of some dynamic system with a scalar vector sequence, i.e. with a set of z-vectors. Then, according to Takens theorem, there is some representation $\Psi(z)$, such that $z_{i+1} = \Psi(z_i)$ or $x_{i+k} = F(z_i)$, where k corresponds to the unknown coordinate of vector z_{i+1}. As a result, it is possible to say that all dynamic methods of analysis and forecasting are actually based on different methods of approximation of one of the forms of this representation. The problems of dynamic analysis and forecasting emerged long before synergy and non-linear dynamics, and have been referred to as the problems of "random processes forecasting" [153, 154]. In practice it was always turning into creation of functions Ψ or F, which approximated the history of time series best. Among the key contributions that non-linear dynamics did for analysis and forecasting are:

- A strong mathematical justification for the idea of time series analysis and forecasting from the point of view of dynamics (Takens theorem) was proposed;
- Based on Takens theorem and the idea of MIM, it became possible to impose a number of limitations on the functions Ψ or F, since a representation exists only on some surface, not in all R^m.

It has to be remembered that dynamic methods of analysing and handling time series have some limitations, and thus are suitable for particular tasks only. At the same time, it is believed that concepts, ideas and approaches to the use of methods of non-linear dynamics are mathematically justified, and are applicable and very useful for this research.

4.5 Analysing and Handling Time Series, Generated by Systems with Fractional Structure

The existence of "statistical fractionality" of real financial markets, noted firstly by Mandelbrot, and supported lately by numerous experimental findings on the existence of fractional structure in the evolution of currency exchange rates, stimulates the consideration of methods for analysing time series, generated by systems with a fractional structure. Firstly, we consider the methods of dimension estimation of these systems by time series. There are a number of dimension, which are used classify some characteristics of the system. Furthermore, several methods for the calculation of one and the same parameters exist for each dimension. Which dimension to choose? Why numerous methods of estimation are needed, and why it is not enough to have only one? The existence of numerous methods of dimension estimation is important. While analysing or reconstructing data by time series, some distortions or other traits, caused by the system, are possible. These distortions or traits result in systematically biased results, and the existence of multiple methods of dimension estimation helps to find the real dimension estimates and to evaluate their accuracy. First attempts at dimension estimation have been made through the capacity d_C [7, 37, 38, 39]. Quantitative methods followed the introduced definition, except the stage of minimisation of the number of covering manifolds. These algorithms became known as *box counting methods*, and the first quantitative estimates of strange attractors' dimensions have been made with their help. However, it soon became evident that this type of algorithm has some serious weaknesses. First of all, in these algorithms it is difficult to formalise the choice of line intervals, that are used for dimension estimation. Secondly, for obtaining good estimates for even simple systems, very large samples are needed (with 10^6 points and more). Smaller samples do not allow fair line intervals to be obtained. Thirdly, it is not always possible to find good estimates of the parameters, since sometimes rarely attended regions of the attractor have significant impact on these parameters and their estimates. When some of these regions are inaccessible, estimates are understated. Due to this, and to a number of other reasons, these algorithms rarely provide reliable estimates. In practice there could be non-convergence of the parameters. As a result, some researchers [7], do not recognise algorithms like box counting as practically applicable, and consider other methods for finding dimensions.

With the application to fractional structures, these methods firstly comprise spectral dimensions [7, 37, 38, 39, 109]. To justify the choice of these methods, it is possible to refer to Schroeder [38], who argues that statistics plays a key role among application fields, where self-similarity exponential methods are applied. Among the most typical homogeneous power methods of the type $f^{-\beta}$ are power spectrums (squares of amplitudes of Fourier transforms), usually referred to as noises. Other researchers support this point of view as well [7, 37, 39, 109]. In other words, lots of real events, processes and signals have one important common property: in some wide frequency range their power spectrums represent homogeneous power functions of type $f^{-\beta}$, where β lies between 0 and 4 [38]. Besides, it has been shown that while considering Brownian motion, real objects can be simulated and then described with fractional descriptor. This, in turn, gives a common Power Spectrum Method [122, 123], allowing to find dimension of the fractal from Fourier power spectrum relatively easily not only for the signal (one-dimensional time series), but also for representation (multy-dimensional time series).

A Fourier transform allows for the representation of some function $X(t)$ in the frequency domain, as the sum of the components with frequency f. Also the function $X(t)$ can be presented as a decomposition of type $\exp(2\pi i f t) = \cos(2\pi f t) + i\sin(2\pi f t)$. The frequency of these functions equals f periods per unit of time, and consequently the period equals $1/f$. The complex notation $\exp(2\pi i f t)$ simplifies the analysis, but since an alternative notation is equivalent to $\cos(2\pi f t)$ and $\sin(2\pi f t)$, calculations can be also made using real numbers. The component

$X(t)$ for frequency f is presented with $\hat{X}(f)\exp(2\pi i f t)$, where $\hat{X}(f) = \int_{-\infty}^{+\infty} X(t)\exp(-2\pi i f t)dt$. The function $\hat{X}(f)$ is usually referred to as the direct Fourier transform of the function $X(t)$. Under certain conditions, the inverse Fourier transform $X(t) = \int_{-\infty}^{+\infty} \hat{X}(f)\exp(-2\pi i f t)df$ is possible. It describes a signal synthesis from separate frequency components. Both the direct and the inverse Fourier transforms have discrete analogies. It is hypothesised that the full power of signals $X(t)$ and $\hat{X}(f)$ is finite, and can be correspondingly presented as $\int_{-\infty}^{+\infty}[X(t)]^2 dt$ and $\int_{-\infty}^{+\infty}[\hat{X}(f)]^2 df$. Moreover, it has been proven [120], that the equality $\int_{-\infty}^{+\infty}[X(t)]^2 dt = \int_{-\infty}^{+\infty}[\hat{X}(f)]^2 df$ exists, which implies that it is possible to find the full power in both time and frequency domains [37]. For a fractional (self-similarity) process $X(t)$ (e.g. describing FBM), it is possible to introduce some function $X(t,T)$, which equals $X(t)$ for $0 \le t \le T$, and equals zero outside the interval $[0,T]$. Then the direct Fourier transform for $X(t,T)$ is $\hat{X}(f,T) = \int_0^T X(t)\exp(-2\pi i f t)dt$, and the mean power of the function $X(t)$ on the interval $[0,T]$ is $\frac{1}{T}\int_0^T[X(t,T)]^2 dt = \frac{1}{T}\int_0^T[\hat{X}(f,T)]^2 df$. As a result, the Power Spectral Density (PSD) of the function $X(t,T)$ will be equal to $M_X(f,T) = \frac{1}{T}[\hat{X}(f,T)]^2$, which allows the PSD of the function $X(t,T)$ in the form of a limit $M_X(f) = \lim_{T\to+\infty}\frac{1}{T}[\hat{X}(f,T)]^2$, for $T \to \infty$. It is also important that one of the basic theorems [120] states that this PSD increase as a function of the frequency, i.e. $M_X(f) \sim f^{-\beta}$, where β is the spectral index, characterising the power spectral dimension of the system. The last expression represents the basis for Fourier Power Spectrum Methods, and allows for rather simple algorithms of finding β for any time series, generated by the systems with fractional structure (fractional signal), when estimation of β reduces to linear approximation of the type $\log M_X(f) \cong -\beta\log(f) + c$ via OLS estimation or with the use of other methods.

With regards to the systems being considered, another important fact is the existence of a simple relation between β and the Hurst parameter H in the form [37, 38]:

$$\beta = 2H + 1. \tag{4.3}$$

Taking into account that $\beta = 2q$, equation (4.3) allows the relationship between the Hurst parameter H and the spectral parameter q to be found. The parameter q arises in equation (3.7) and characterises the spectral dimension of the amplitude of the Fourier transform for the system in the form:

$$\frac{\beta}{2} = q = H + ½ . \tag{4.4}$$

Firstly, equations (4.3) and (4.4) allow the time series to be analysed with two different methods, and then allow a compare of the results. This could help in finding the real dimensions and estimation errors. Secondly, taking into account the ratio $d = 2 - H$, it becomes possible to estimate the dimension of the realisation, sometimes called Fourier dimension. Thirdly, these equations allow the methods of non-linear dynamics to be applied, and for analysing not only the general behaviour of the time series (for $T \to \infty$), or finding its behaviour on some finite interval $[0,T]$, but also for obtaining such important characteristics of the system, as $q(t,T)$ and $H(t,T)$, and for testing the model (3.7). Finding and describing the behaviour of the parameter $q(t,T)$ can be considered as one of the key research interests of this study. This main focus of the research is both justified and very perspective since the parameter $q(t,T)$ can be considered not only as spectral parameter, included into fractional equations, or macroeconomic parameter, but also as an independent object of dynamic analysis, characterising multifractional properties of the system. It is allows the simulation of the dynamics of the process for real price characteristics to be simplified, e.g. using the method of Fourier filtration [37].

When considering a real time series it is useful to apply diffusion models based of Brownian motion, including FBM. Using the properties of the FBM, for increments and the variance of increments it is possible to write [37]:

$$\mathrm{E}|X(t_2) - X(t_1)| = \sqrt{\frac{2}{\pi}}\sigma(t_2 - t_1)^H \text{, and} \tag{4.5}$$

$$\mathrm{E}(X(t_2) - X(t_1))^2 = \sigma^2 \,|\, t_2 - t_1 \,|^{2H} . \tag{4.6}$$

By combining equations (4.5) and (4.6), it is possible to find a simple algorithm for finding H and for comparing the results. Also it becomes possible to analyse such parameters as changes in volatility, the scope of deviation from normality, etc.

This analysis would be incomplete if the probabilistic dimensions D_q are an omitted, while infinite continuum set of probabilistic dimensions, proposed by Rennie, includes many of the recognised dimensions. By definition, Rennie's q-dimension can be obtained with the following formula [7, 37, 38, 39, 109]:

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{i=1}^{N} p_i^q}{\log \varepsilon}, \tag{4.7}$$

where p_i can be interpreted as the probability of getting into fractal's component i. Generally $-\infty < q < +\infty$, but from the point of view of practical use, we would not be interested in the whole interval of dimension. Once we are focused mostly on the estimation of essential variables and on the analysis of the main properties of the system, negative values of q can be excluded, and only one dimension need be examined. In particular, for $q = 0$ we get the dimension d, for $q \to 1$ after eliminating uncertainty, we get the so-called *information dimension*. For $q = 2$, equation (4.7) gives the so-called *correlation index* or *correlation dimension*:

$$D_2 = \lim_{\varepsilon \to 0} \frac{\log \sum_{i=1}^{N} p_i^2}{\log \varepsilon} \tag{4.8}$$

Correlation dimension has one important practical advantage – it can be relatively easy obtained from the available experimental data. The theoretical importance of dimension D_2 is stipulated by its close relation with such a fundamental concept, as correlation. Dimension D_2 is directly determined with the correlation function of the fractional manifold, i.e. by the probability of finding elements, belonging to this manifold, but distanced from each other. As a result, finding the correlation dimension D_2 simplifies significantly. For this research we will only use dimension D_2. This choice is well justified because the correlation dimension and corresponding correlation integral have numerous extra virtues. In particular, they allow for the estimation of other characteristics of the system, e.g.: entropy and noise level, or, what is more important for us – distinguish chaotic systems from stochastic [7, 38]. At the heart of finding D_2 is the calculation of generalised entropies: $H_q = \frac{\log \sum_{i=1}^{N} p_i^q}{1-q}$, where p_i is the measure of block i, covering the attractor. For $q > 1$, the main contribution to the sum under the logarithm is made from the most frequently attended blocks, while the impact, that rarely attended blocks play, is going down. Thus, it becomes possible to solve one of the problems of methods like box counting, in which the contribution made to the dimension d from rarely used regions of the attractor is significant. Using the same fragmentation of blocks, it is possible firstly to find $H_q(\varepsilon)$ for each block, then similarly to the capacity d, to estimate D_q via the linear approximation $H_q(\varepsilon) \cong -D_q \log \varepsilon + c$. However, this approach still requires large samples. The following idea changes the situation crucially. The sum $C_q(\varepsilon) = \left(\sum p_i^q\right)^{\frac{1}{q-1}}$, often called the *generalised correlation integral*, can be presented as: $C_q(\varepsilon) = \left(\sum p_i^{q-1} p_i\right)^{\frac{1}{q-1}} = \left\langle p_i^{q-1} \right\rangle^{\frac{1}{q-1}}$, i.e. as a (geometrical) mean of the measure of the block of size ε. The idea is to find the best estimation of this mean by finding the measure of the block or sphere of radius ε with its centre at point x_i, and then average out the results for all these spheres. Let us denote with $k_i(\varepsilon)$ the number of points x_j inside the ε - sphere with the centre at point x_i. Its measure can be approximately estimated as $P_i(\varepsilon, N) = \frac{k_i(\varepsilon)}{N}$, and $C_q(\varepsilon) = \left(\frac{1}{N}\sum P_i(\varepsilon)^{q-1}\right)^{\frac{1}{q-1}} = \left(\frac{1}{N}\sum\left(\frac{k_i(\varepsilon)}{N}\right)^{q-1}\right)^{\frac{1}{q-1}}$. In the case when $q = 2$, this equation simplifies significantly to $C_2(\varepsilon) \equiv C(\varepsilon) = N^{-2}\sum k_i(\varepsilon)$. Value C_2 is often called a *correlation sum*, or simply a *correlation integral*. The last sum is just double the number of pairs of points, distance from each other by no more then ε. We come to the widely recognised definition of the correlation integral: $C(\varepsilon) = \frac{\textit{the number of pairs with } |x_i - x_j| < \varepsilon}{\textit{the number of pairs with } x_i, x_j}$. Then the correlation dimension is defined as:

$$D_2 = \lim_{\varepsilon \to 0} \frac{C(\varepsilon)}{\log \varepsilon}. \tag{4.9}$$

For finding the dimension D_2 from equation (4.9), smaller samples are required (in comparison with the previous approaches), though the estimation methodology remains the same – linear approximation of $C(\varepsilon) \cong -D_2 \log \varepsilon + c$ with the "most linear" section. This algorithm was proposed by Grassberg and Procaccia [155], and since then it has been probably the most popular algorithm of non-linear dynamics for time series. However, all the benefits of this algorithm collide with massive number of calculations – $O(N^2)$ is required. Therefore, further research was aimed at the development of effective algorithms of its realisation [156, 157, 158].

Turning to the question of distinguishing chaotic and stochastic sequences, it has to be remembered that purely determined dynamic systems can exhibit properties of stochastic white noise [6, 7, 11, 19, 22, 28, 36,..., 39, 109]. Problems of distinguishing chaotic and stochastic sequences are vital, not only in terms of this dissertation. Answers to these questions allow for an understanding of the nature of the irregularities in financial data. There are a number of approaches for doing this [6, 38], and in one of them [92, 159], the key role in distinguishing chaotic and stochastic nature is played by the correlation dimension and the correlation integral, defining the function: $C(\varepsilon)=\lim_{N\to+\infty}\frac{k(N,\varepsilon)}{N^2}$, where $k(N,\varepsilon)$ is the number of those pairs (i,j), $i,j\le N$, for which $|x_i-x_j|<\varepsilon$ in the sequence (x_n). In addition to the function $C(\varepsilon)$, another function is also used: $C_m(\varepsilon)=\lim_{N\to+\infty}\frac{k_m(N,\varepsilon)}{N^2}$, where $k_m(N,\varepsilon)$ is the number of those pairs (i,j), for which all components of the vectors $(x_i,x_{i+1},...,x_{i+m-1})$ and $(x_j,x_{j+1},...,x_{j+m-1})$, $i,j\le N$, differ by less than ε.

For stochastic sequences (x_n) of a white noise type, for small ε, $C_m(\varepsilon)\sim\varepsilon^{V_m}$, where the fractional parameter $V_m=D_2$. This property is typical of most of the determined systems, e.g. of logistic systems [92]. The idea of a method for distinguishing chaotic and stochastic sequences is based on the fact that the correlation dimensions of these sequences are different [6, 38, 92, 159]. For stochastic sequences it is greater, and for chaotic sequences it is smaller. In this case the method of non-linear dynamics, based on Takens theorem, is applied: for some set of measures equidistant in time, *m*-dimensional vectors are constructed, and then their correlation dimension D_2 is determined. If estimated parameters are really random, then with increasing dimension of reconstruction (enclosure) *m*, the determined correlation dimension also has to increase. But for chaotic determined system (regardless how chaotic it looks), dimension D_2 stops increasing after the correlation dimension of D_2 becomes smaller than dimension of the enclosure *m* [38]. The method of determining fractional dimension of experimental data by enclosing it into the space with corresponding dimension, and allowing for distinguishing the determined chaos from random noise, was successfully applied to numerous physical, meteorological and physiological observations [38]. With regards to financial time series, this method is examined in the work of Pesaran and Potter [92], with corresponding econometrical comments provided. Therefore, the choice of these methods for this research is well justified.

Also let also pay attention to other methods of distinguishing chaotic and stochastic sequences. Let $x=(x_n)$ be a chaotic sequence, caused by some dynamic system with probability distribution $F=F(x)$ for x_0, invariant to this system. Let $\tilde{x}=(\tilde{x}_n)$ be a stochastic sequence, consisting of IID parameters with an univariate distribution $F=F(x)$. Let us organise the parameters $M_n=\max(x_0,x_1,...,x_n)$ and $\tilde{M}_n=\max(\tilde{x}_0,\tilde{x}_1,...,\tilde{x}_n)$, and let also $F_n(x)=P(M_n\le x)$ and $\tilde{F}_n(x)=P(\tilde{M}_n\le x)$. The idea of this approach, proposed in [160], is based on the observation that a maximum is a good characteristic for finding differences between chaotic and stochastic sequences. In [160] it is shown that for most dynamic systems, the behaviour of $F_n(x)$ is different from the behaviour of $\widetilde{F}_n(x)$. Thus, it is possible to consider the maximum as a good statistic for distinguishing chaotic and stochastic sequences in some models. Other perspective methods of distinguishing chaotic and stochastic sequences are based on the use of Fourier techniques [120], however, their practical application is still under consideration.

4.6 CONCLUSION

Taking into account the nature of this research, it is worth focusing on finding solutions to the problem of identification of time series. Statistical methods of analysing and handling time series, including different statistical models, *R/S Analysis* and OLS, have to be considered not only as basic mathematical and statistical tools, but as the key sources for obtaining new information about research objects. Statistical *R/S Analysis* turns out to be very helpful for finding after-effects, memory effects, persistence and antipersistence, but also allows us to reveal periodic and non-periodic cycles in the considered time series. Choosing of optimal observed parameters, and the methods of finding the interrelations among them, can be considered as one of the main tasks in this research. For small coefficient of correlation the OLS estimate is ineffective, and alternative methods are required. Methods of finding and analysing time series using non-linear dynamics indicate a new research field, that focuses on the construction and analysis of various phase diagrams, estimation of dimension of the dynamic system and its attractor in combination with the reconstruction of the attractor from the experimental data. While analysing time series (generated from systems with fractional structure) in combination with conceptual approaches of non-linear dynamics, Fourier techniques (including Fourier transform); general Power Spectrum Method; and Fourier filtration; etc. need to be considered. Many actual problems of distinguishing chaotic and stochastic sequences remain open. At present, to solve these problems there are a number of approaches. In one of them, an important role is played by the correlation dimension and the correlation integral. The possibility of using the correlation dimension with the enclosure of these data into the space of appropriate dimension is well justified for this dissertation. Besides, the main research interest of this study, aimed at finding and describing behaviour of parameters $q(t,T)$, is justified and rather perspective since parameters $q(t,T)$ can be considered not only as spectral parameter, included into fractional equations, or some macroeconomic indicator, but also as an independent object of dynamic analysis, characterising multifractionality of the system, what allows for simplification and enlarging of the processes of analysing and simulation of the real time series.

CHAPTER V. Theoretical Analysis of the Evolution of the Spectral Parameter for Time Series of Currency Exchange Rates

5.1 INTRODUCTION

This chapter presents the examination of the set of parameters, determining the characteristics of the evolution process of the spectral parameter q for currency exchange rates. For this, a kinetic equation of the evolution of the spectral parameter is empirically obtained, and a theoretical model for the distribution function $f(q,t)$, characterising the change in time of the density probability of the parameter q, is proposed. The processes of change of the spectral parameter were statistically simulated (including the use of asymptotic methods), and some parameters, determining the shape of the distribution, were obtained.

5.2 FINDING THE KINETIC EVOLUTION EQUATION OF THE SPECTRAL PARAMETER OF TIME SERIES OF CURRENCY EXCHANGE RATES

Currency exchange rates depend upon many parameters, which we will consider as the components of some generalised vector $\vec{a}$. Then, for the evolution of some i realisation of the exchange rate (i.e. for some market participant i), the domain of the variation of the multidimensional vector $\vec{a}$ will constitute a multidimensional space A_i. Generally, for various realisations of exchange rate, the size of the space can be different because some of the values could be irrelevant or forbidden. Then the total population of components of all vectors $\vec{a}$ of all realisations of exchange rate will constitute some generalised phase space A. Let us also consider N realisations of some currency exchange rate (there are N participants in the market) of the system, and let the vector $\vec{a}$ for each realisation consisting of N_m components. Then the space A_i will be N_m-dimensional, and the space A will be $(N_m \times N)$-dimensional. In this case, for N realisations the state of the system at the micro level is defined by a fixed set N_m. This set can be presented by means of some affix (phase point) in the generalised phase space A. For a continuously changing state, these affixes, corresponding to different moments of time t, develop a trajectory in the space. Differential equations, describing this trajectory, can be generally represented by the following formula:

$$\frac{d\vec{a}_i}{dt} = f(\vec{a}_1, \vec{a}_2, ..., \vec{a}_N, t), \tag{5.1}$$

where $\vec{a}_i$ is a generalised vector of realisation i. The number of these equations obviously equals $(N_m \times N)$. These equations hold all the information of the evolution process in the system at the micro level. It becomes possible to describe macro level changes in the evolution process with either continuous or piecewise-continuous functions of generalised coordinates. When applying statistical method, we assume that there is a relation between any microscopic function i and some unique macroscopic function.

Let us apply the methods of probabilistic statistical description and consider the prototype of the system instead of the real macrosystem, described with equation (5.1). For this, at the initial moment of time let us choose N_m different macrosystem copies, uncorrelated with each other, and each having exactly N elements. This, for example, corresponds to the groups of market participants with similar aims and investment horizons. All these systems can also be presented as points in the space A. This aggregate of phase points can be realised as a continuum (such as a liquid) in a space A. The aggregate of phase points, describing the state of the macrosystem copies, uncorrelated with each other, is usually referred to as a *statistical assembly*.

Let the number of phase points be very high, and also let the systems to be chosen in such a way, that in the space A it is possible to introduce a continuous function $p(\{A\},t)$, representing the density of points of the describing system. The notion of continuity in a generalised phase space in not trivial and requires clarification. From now we consider only the physical description of a system, achieved through the accuracy of calculations/estimations of parameters, and all the variations of the parameters within the predetermined accuracy as physically indistinguishable. Having defined the accuracy of the components of vector $\vec{a}$, we have also determined the size of some volume ΔA, such that all points, describing systems, which are in ΔA simultaneously, are physically indistinguishable. The continuity of the function $p(\{A\},t)$ in the space A implies the coincidence of values of this function within ΔA, with a given accuracy. This allows us to consider a volume ΔA as infinitesimal ($\Delta A = dA$). Thus, a set of variables and the accuracy of their recorded uniquely define the level of descriptive accuracy of the whole system.

In terms of a statistical approach, it is assumed that at each specified moment of time a macrosystem (with some probability) could be in any possible state, characterised with the corresponding phase points. The probability $dW(\{A\},t)$, that at this moment of time the phase point belongs to the element $\{[A, A+dA]\}$ of the phase space volume, i.e. lies within the neighbourhood of the point $\{A\}$, can be found with the PDF $f(\{A\},t)$. Then the phase point of the macrosystem has coordinates $\{a\}$ at time t: $dW(\{A\},t) = f(\{A\},t)d\{A\}$, where $d\{A\} = \prod_1^N \prod_1^{Nm} da$. Since we assumed that macrosystem copies are uncorrelated with each other, then the probabilities that the phase points of the macrosystem copies belong to the selected interval of the phase space are also independent. Assuming that the number of phase points N_0 in a statistical assembly is high enough to apply the law of large numbers, it is possible to write: $\lim_{N_0 \to \infty} \frac{1}{N_0} \rho(\{A\},t) = dW(\{A\},t)$, where $dW(\{A\},t)$ is the probability that the phase point of any randomly selected macrosystem copy belongs to the selected interval of the phase space at the given moment of time. Then for the statistical assemblies with a sufficiently high number of phase points N_0: $f(\{A\},t) = \frac{\rho(\{A\},t)}{N_0}$. The distribution of the phase points of the macrosystem copies of the statistical assembly in the phase space can be represented by the function $f(\{A\},t)$, since $f(\{A\},t)$ and $\rho(\{A\},t)$ coincide accurately to a constant.

Let us assume that the spaces of change of realisations of some currency exchange rate coincide, i.e. $A_i = A_0$, $i = 1,2,...,N$. We assume that all behaviour characteristics of the system are captured by the spectral parameter q. Then, to describe the system, the kinetic equation of its evolution in terms of q is required. The formal derivation of this equation, implying averaging by the assembly of realisations, goes beyond the research interests of this dissertation, because it is often dependent on unsolvable problems, requiring extensive knowledge of statistical physics, microeconomics and macroeconomics. Let us consider a more simple – phenomenological approach and introduce macroscopic PDFs $f_N(q,t)$ and $f(q,t)$, such that $f_N(q,t) = N_q(t) f(q,t)$, normalised corresponding to the number $N_q(t)$ of experimental points q and to unity, i.e.:

$$\int_{-\infty}^{+\infty} f_N(q,t)dq = N_q(t) \text{ and } \int_{-\infty}^{+\infty} f(q,t)dq = 1. \tag{5.2}$$

Let us define $D_+(q - \Delta q)$ as the parameter characterising the intensity of the apparently irregular process of transition of q from the class with characteristic values $(q - \Delta q)$ to the class with characteristic values (q) per unit time. This process is caused by changes occurring during the recording of value $N_q(t)$. Similarly, let us define $D_-(q)$ as the parameter characterising the intensity of the apparently irregular process of transition of q (during the same process) from the class with characteristic value (q) to the class with characteristic value $(q-\Delta q)$. Then, for a non-stationary state, the right hand side of equation:

$$J(q,t) = f(q-\Delta q,t)D_+(q-\Delta q) - f(q,t)D_-(q), \tag{5.3}$$

characterises the excess/shortage of that set of points, which go from the class $(q-\Delta q)$ in the class (q) per unit of time, over those, which go from the class (q) in the class $(q-\Delta q)$ within the same time. To find the kinetic equation of the macroscopic function $f(q,t)$, we introduce the function of its stationary distribution $f_s(q)$. It is assumed that generally this system could reach a stationary and/or equilibrium state, in which the portion of points with any characteristic parameter q stays almost constant during the process of registration. Correlation between $D_+(q-\Delta q)$ and $D_-(q)$ can be obtained from the condition:

$$f_s(q-\Delta q)D_+(q-\Delta q) = f_s(q)D_-(q) \text{ or } D_+(q-\Delta q) = \frac{f_s(q)D_-(q)}{f_s(q-\Delta q)}. \tag{5.4}$$

Bringing equation (5.4) into (5.3) we get:

$$J(q,t) = f_s(q)D_-(q)\left[\frac{f(q-\Delta q,t)}{f_s(q-\Delta q)} - \frac{f(q,t)}{f_s(q)}\right]. \tag{5.5}$$

Assuming that for any q, Δq is much smaller than q, $J(q,t)$ in equation (5.5) can be treated as the function of the continuous argument q. Therefore: $J(q,t) = -f_s(q)D_-(q)\frac{\partial}{\partial q}\left[\frac{f(q,t)}{f_s(q)}\right]$. Assuming that $D_-(q) = D(q)$,

$$J(q,t) = -D(q)\frac{\partial f(q,t)}{\partial q} + D(q)f(q,t)\frac{\partial \ln f_s(q)}{\partial q}. \tag{5.6}$$

If $\ln f_s(q) = const - k\Delta U(q)$ is taken, equation (5.6) becomes formally identical as the recognised physical equation of photon flux with PDF $f(q,t)$ caused by diffusion, and by some imposed force F. This force F corresponds to the potential energy $\Delta U(q)$, therefore:

$$J(q,t) = -D(q)\frac{\partial f(q,t)}{\partial q} - kD(q)f(q,t)\frac{\partial \Delta U(q)}{\partial q}. \tag{5.7}$$

Then, the intensity of change $-\frac{\partial f(q,t)}{\partial t}$ can be defined as:

$$\frac{\partial f(q,t)}{\partial t} = J(q,t) - J(q+\Delta q,t) = -\frac{\partial J(q,t)}{\partial q}. \tag{5.8}$$

Combining equations (5.7) and (5.8), we get the required kinetic equation of the process of evolution of q, obtained phenomenologically:

$$\frac{\partial f(q,t)}{\partial t} = k\frac{\partial}{\partial q}\left[D(q)f(q,t)\frac{\partial \Delta U(q)}{\partial q}\right] + \frac{\partial}{\partial q}\left[D(q)\frac{\partial f(q,t)}{\partial q}\right]. \quad (5.9)$$

The equation (5.9) allows the process of change of q to be considered as some random process, equivalent to some physical transport process. Under $\langle w(q)\rangle = -kD(q)\frac{\partial \Delta U(q)}{\partial q}$, where $\langle w(q)\rangle = \frac{dq}{dt}$ is the average speed of the systematic change of q, equation (5.9) coincides with the special case of the known FPK equation (for $D(q,t) = D(q)$, $\langle w(q,t)\rangle = \langle w(q)\rangle$). This equation is also called the Fokker-Planck equation.

The stationary distribution $f_s(q)$ can be obtained from equation (5.9) when $\frac{\partial f(q,t)}{\partial t} = -\frac{\partial J(q,t)}{\partial q} = 0$, which is possible when $J(q,t) = const$. Taking into account that the characteristic parameter q stays almost constant during the process of recording, we get: $J(q,t) = -D(q)\frac{\partial f(q,t)}{\partial q} - kD(q)f(q,t)\frac{\partial \Delta U(q)}{\partial q} = 0$ or $\frac{\partial f_s(q)}{\partial q} = -kf_s(q)\frac{\partial \Delta U(q)}{\partial q}$. After integration we get:

$$f_s(q) = K\exp(-k\Delta U(q)). \quad (5.10)$$

Similar to the physical processes, the function $f_s(q)$ is defined with system's "energy" $\Delta U(q)$. Using maximum entropy (a recognised physical principle), and assuming that equilibrium and stationary states coincide, it is possible to state that the distribution (5.10) corresponds to the system, in which, besides (5.2), there is one more constraint: $\int_{-\infty}^{+\infty}\Delta U(q)f_s(q)dq = \Delta U_{av}(q)$, where $\Delta U_{av}(q)$ is the average "energy" of the system. We obtain the canonical distribution of the closed macrosystem, showing that (under the assumptions made) there are no limitations on the function type of $f_s(q)$, except the normalisation condition and the requirement for a fixed average "energy" of the system. Shortly after the process starts, the elements of such system are most likely to be in the position, corresponding to distribution (5.10).

The stochastically determined nature of the processes occurring in the FX-market usually exhibits the existence of profit fluctuations, corresponding to the rate of change of the determining parameters. We will not examine formal laws of microeconomics and macroeconomics, deliberately substituting them with no less formal probabilistic laws. Assume, that all changes in the FX-market appear in the existence of fluctuations $\tilde{w}$ of speed $w(q)$ for the parameter q for time series of currency exchange rate. We also assume that fluctuations occur around the average (for a given q) value of the speed $\langle w(q)\rangle$, which results to fluctuations $\tilde{f}$ of the macroscopic PDF $f(q,t)$ around its mean $\langle f(q,t)\rangle$. It is possible to write that: $w(q) = \langle w(q)\rangle + \tilde{w}$

and $f(q,t) = \langle f(q,t)\rangle + \tilde{f}$, where $\langle w(q)\rangle = \frac{dq}{dt}$ is the average speed of the systematic change of q; $\tilde{w}$ is the fluctuating component of the speed $w(q)$; $\langle f(q,t)\rangle$ is the mean PDF; and $\tilde{f}$ is the fluctuating component of $f(q,t)$. Let us also make the following assumptions on the nature of $\tilde{w}$: the fluctuating component is time independent and changes very quickly in relation to changes in q; and there is a time interval Δt, during which changes in q are rather small.

Taking $w(q)f(q,t) = J(q,t)$, and similarly to equation (5.8), it is possible to write: $\frac{\partial f(q,t)}{\partial t} = J(q,t) - J(q+\Delta q,t) = -\frac{\partial J(q,t)}{\partial q} = -\frac{\partial}{\partial q}[w(q)f(q,t)]$. Taking into account fluctuations, we get: $\frac{\partial}{\partial t}\left[\langle f(q,t)\rangle + \tilde{f}\right] = -\frac{\partial}{\partial q}\left[\langle w(q)\rangle\langle f(q,t)\rangle + \langle w(q)\rangle\tilde{f} + \tilde{w}\langle f(q,t)\rangle + \tilde{w}\tilde{f}\right]$, or, after averaging over some finite time interval, $\frac{\partial}{\partial t}[\langle f(q,t)\rangle] = -\frac{\partial}{\partial q}\left[\langle w(q)\rangle\langle f(q,t)\rangle + \langle \tilde{w}\ \tilde{f}\rangle\right]$. From the definitions of $\tilde{w}$ and $\tilde{f}$, it follows that their mean values $\langle\tilde{w}\rangle$ and $\langle\tilde{f}\rangle$ are equal to zero. However, $\langle \tilde{w}\ \tilde{f}\rangle$ should not necessarily vanish, as there could be a correlation between local values of the pulsating components, caused by the non-linearity of $w(q)$ and $f(q,t)$. Introducing $\langle \tilde{w}\ \tilde{f}\rangle$ as $\langle \tilde{w}\ \tilde{f}\rangle = -\frac{\partial}{\partial q}[D(q)\langle f(q,t)\rangle]$, then for $\langle f(q,t)\rangle$:

$$\frac{\partial f(q,t)}{\partial t} = -\frac{\partial}{\partial q}[\langle w(q)\rangle f(q,t)] + \frac{\partial^2}{\partial q^2}[D(q)f(q,t)], \tag{5.11}$$

where $D(q)$ is the diffusion coefficient on the set of parameters q. We again get the Fokker-Planck equation, but now using phenomenologically-probabilistic assumptions. It is possible to consider that the existence of fluctuations $\tilde{w}$ of the speed $w(q)$ for parameters q can be taken into account in equation (5.11) through the choice of values of the diffusion coefficient $D(q)$ in the space of the parameter q. So, we found not only the necessary kinetic equations of the processes in the forms (5.9) and (5.11), but also obtained the constraints for the application of these equations, and found a correlation between their components. The equations (5.9) and (5.11), describing the evolution of any initial density distribution of the spectral parameter q, are special cases of the recognised kinetic FPK equation. Equation (5.11) has a strong probabilistic statistical basis, and can be applied to study a wider range of macrosystems, where the processes of change of the parameters in time are Markovian processes. Following this discussion, a review of the Markovian processes will provide an insight into the correlation of the parameters, as well as the motives and the nature of the processes, described by equations (5.9) and (5.11).

5.3 PROPERTIES OF A PRACTICAL APPLICATION OF THE EVOLUTION EQUATION OF THE SPECTRAL PARAMETER

Let the time series of a some currency be characterised by the set of spectral parameters $q_1, q_2, ..., q_n$, and q can be viewed within time interval $t = 1,2,...$, which can be represented as a stepwise change from one state into another. Then the set of system states is a *discrete Markov chain* if the probability that during an observation t the system is in the k^{th} state (q_k) is completely determined with the state requirements (q_l) from one of the previous observations $t_0 < t$. This probability can be written as $p(q_k, t; q_l, t_0)$, and can be considered as a transition probability from state q_l to state q_k in the period between t_0 and t. This implies that the transition probability does not depend on the state of the system before time t_0. In other words, the evolution process of q takes every new step without any information on how the present state has been reached, and thus the probability of the system's present state is thoroughly dependent on its previous state. From this point of view, it is possible to say that the system has no memory in space q. This does not necessarily imply that currency exchange rates have no memory, since its existence is captured by the parameter q. These circumstances impose a number of constraints on the type of function p, which we will be looking at in the example of a continuously changing system.

In case of continuous change it is possible to assume that there is a link between consecutive events within a relatively small time interval. If at time t_0, a variable takes on the value q_0, this affects the probability of a variable taking on value q at time t. The probability of finding the value of q within an infinitesimal interval from q to $q + dq$ at time t equals to $f(q,t)dq$. For a Markovian random process, the link between the two consecutive events is estimated through the probability of transition. Instead of transition probabilities $p(q,t;q_0,t_0)$ it is possible to use the function $f_e(q,t \mid q_0,t_0)$, defining a density probability of a macrosystem being in state q at time t, if at $t_0 < t$ the system was in state q_0. The PDF $f_e(q,t \mid q_0,t_0)$ can be considered as the analogue of conditional probabilities or conditional correlation functions. With the function $f_e(q,t \mid q_0,t_0)$ it is possible to find a distribution function $f(q,t)$, defining the probability of different states of the system at time t:

$$f(q,t) = \int_{-\infty}^{+\infty} f_e(q,t \mid q_0,t_0) f(q_0,t_0) dq_0 . \tag{5.12}$$

Here $f(q_0,t_0)dq_0$ defines probabilities of different values of the observed parameter at an initial moment of time and $f_e(q,t\,|\,q_0,t_0)dq$ defines the transition probability of the macrosystem from the state q_0 at time t_0 to the states $[q;q+dq]$ at time t. The function $f_e(q,t\,|\,q_0,t_0)$ cannot be negative, and has to satisfy the normalisation requirement: $\int_{-\infty}^{+\infty} f_e(q,t\,|\,q_0,t_0)dq = 1$. For Markovian processes, transitions $(q_0,t_0) \to (q_1,t_1)$ and $(q_1,t_1) \to (q,t)$ are independent (δ-correlated with time). Then, considering the transition $(q_0,t_0) \to (q,t)$, as a superposition of all feasible transitions $(q_0,t_0) \to (q_1,t_1) \to (q,t)$, it becomes possible to get the following equation for the function $f_e(q,t\,|\,q_0,t_0)$:

$$f(q,t\,|\,q_0,t_0) = \int_{-\infty}^{+\infty} f_e(q,t\,|\,q_1,t_1) f_e(q_1,t_1\,|\,q_0,t_0)dq_1 . \tag{5.13}$$

In mathematics integral equation (5.13) is called the Chapman-Kolmogorov equation for examining transition probabilities. It is possible to calculate the transition probability from state q_0 at time t_0 to state q at time t, if the product of the transition probability from q_0 at time t_0 to some q_1 at any time t_1, and then from this q_1 to q at time t is taken, and summed for all possible values of q_1. Multiplying equation (5.13) by $f(q_0,t_0)$ and integrating by all possible values of q_0 (taking into account equation (5.12)), we get:

$$f(q,t) = \int_{-\infty}^{+\infty} f(q_1,t_1) f_e(q,t\,|\,q_1,t_1)dq_1 . \tag{5.14}$$

The integral equation (5.14) is a more general form of the equation, describing a change in the function $f(q,t)$ during a Markovian process. Let us find conditions, under which equation (5.14) can be reduced to (5.9) and (5.11). For simplicity, we consider only Markovian processes homogeneous in time, where the function $f_e(q,t\,|\,q_0,t_0)$ can be presented in the form:

$$f_e(q,t\,|\,q_0,t_0) = f_e(q,q_0,t-t_0). \tag{5.15}$$

Substituting equation (5.14) with the right hand side of equation (5.15) we get:

$$f(q,t) = \int_{-\infty}^{+\infty} f(q_0,t_0) f_e(q,q_0,t-t_0)dq_0 . \tag{5.16}$$

For convenience let us introduce new parameter $\xi = q - q_0$, and instead of the function $f_e(q,q_0,t-t_0)$ let us use the equivalent function $\varphi(q-\xi,\xi,t-t_0)$, defining transition probabilities $q-\xi \to q$. Then, instead of equation (5.16) it is possible to get:

$$f(q,t) = \int_{-\infty}^{+\infty} f(q-\xi,t_0)\varphi(q-\xi,\xi,t-t_0)d\xi . \tag{5.17}$$

This process will be considered as "slow" if for a given $t-t_0$, the function $\varphi(q-\xi,\xi,t-t_0)$ takes on relatively high values only for small ξ, i.e. most probable are those transitions $q-\xi \to q$, where the values of the change ξ in the parameter are small. This assumption is equivalent to the statement that integral equation (5.17) is significantly different from zero only for $\xi \approx 0$. Let us use Taylor decomposition for the function $\Phi \equiv f(q-\xi,t_0)\varphi(q-\xi,\xi,t-t_0)$ with the argument $\eta = q-\xi$ around the point $\eta = q$, and focus only on the first few items of this decomposition:

$$f(q-\xi,t_0)\varphi(q-\xi,\xi,t-t_0) \approx$$
$$\approx f(q,t_0)\varphi(q,\xi,t-t_0) - \xi\left(\frac{\partial \Phi}{\partial \eta}\right)_{\eta=q} + \frac{\xi^2}{2}\left(\frac{\partial^2 \Phi}{\partial \eta^2}\right)_{\eta=q} - \frac{\xi^3}{6}\left(\frac{\partial^3 \Phi}{\partial \eta^3}\right)_{\eta=q} \tag{5.18}$$

Substituting equation (5.17) with the right hand side of equation (5.18), and taking into account the normalisation requirement $\int_{-\infty}^{+\infty} \varphi(q,\xi,t-t_0)d\xi = 1$, it is possible to get:

$$f(q,t) - f(q,t_0) = \int_{-\infty}^{+\infty} \left\{-\xi\frac{\partial}{\partial q} + \frac{\xi^2}{2}\frac{\partial^2}{\partial q^2} - \frac{\xi^3}{6}\left(\frac{\partial^3 \Phi}{\partial q^3}\right)\right\} \times f(q,t_0)\varphi(q,\xi,t-t_0)d\xi . \tag{5.19}$$

Introducing $\Delta t = t - t_0$ and $\langle \xi^n \rangle_{\Delta t} = \int_{-\infty}^{+\infty} \xi^n \varphi(q,\xi,\Delta t)d\xi$, equation (5.19) can be written as:

$$f(q,t) - f(q,t_0) = -\frac{\partial}{\partial q}\left[\langle \xi \rangle_{\Delta t} f(q,t_0)\right] + \frac{1}{2}\frac{\partial^2}{\partial q^2}\left[\langle \xi^2 \rangle_{\Delta t} f(q,t_0)\right] - \frac{1}{6}\frac{\partial^3}{\partial q^3}\left[\langle \xi^3 \rangle_{\Delta t} f(q,t_0)\right]. \tag{5.20}$$

Taking a limit of equation (5.20) for $\Delta t \to 0$, we get the following equation for the function $f(q,t)$: $\frac{\partial f(q,t)}{\partial t} = -\frac{\partial}{\partial q}[A(q)f(q,t)] + \frac{1}{2}\frac{\partial^2}{\partial q^2}[B(q)f(q,t)] - \frac{1}{6}\frac{\partial^3}{\partial q^3}[C(q)f(q,t)]$, where the functions $A(q)$, $B(q)$, $C(q)$ are defined as:

$$A(q) = \lim_{\Delta t \to 0} \frac{\langle \xi \rangle_{\Delta t}}{\Delta t}, \tag{5.21}$$

$$B(q) = \lim_{\Delta t \to 0} \frac{\langle \xi^2 \rangle_{\Delta t}}{\Delta t}, \tag{5.22}$$

$$C(q) = \lim_{\Delta t \to 0} \frac{\langle \xi^3 \rangle_{\Delta t}}{\Delta t}. \tag{5.23}$$

Thus, finite limits (5.21) ... (5.23) represent the first condition, under which integral equation (5.14) can be reduced to differential equations of the type (5.9) or (5.11). If during the time Δt, the parameter changes from q_0 to q, then $A(q) = \lim_{\Delta t \to 0} \frac{\langle q - q_0 \rangle_{\Delta t}}{\Delta t}$ is the average speed of the systematic change of q, which is finite and different from zero. At the same time, $B(q) = \lim_{\Delta t \to 0} \frac{\langle (q-q_0)^2 \rangle_{\Delta t}}{\Delta t}$ can be treated as a local speed of change of variance of q, i.e. as a measure of irregularity of change of the observed parameter (of fluctuations' intensity) within the same interval Δt, because $(q-q_0)^2$ is independent of direction of change of q in relation to q_0. Equation (5.22) also represents the assumption that $\langle(q-q_0)^2\rangle$ is proportional to Δt under small Δt. This assumption supports the fact that for small Δt, probabilities of any significant changes in q are very small and quickly tend to zero. This condition can be represented as $C(q)\to 0$. If for small $(q-q_0)$, the parameter $(q-q_0)^3$ is very small, and for large $(q-q_0)$, the value $\varphi(q,\xi,\Delta t)$ tends to zero so quickly that the product $\xi^n\varphi(q,\xi,\Delta t)$ tends to zero together with Δt, but in such a way that $B(q)\neq 0$. Changes in q, occurring within some time intervals, and caused by a fluctuating component, are statistically independent for non-overlapping time intervals. Thus, the conclusions drawn under this condition are inapplicable for very small Δt, which allows the character of identity fluctuations to be analysed. Taking into account these conditions and constraints, we obtain:

$$\frac{\partial f(q,t)}{\partial t}=-\frac{\partial}{\partial q}[A(q)f(q,t)]+\frac{1}{2}\frac{\partial^2}{\partial q^2}[B(q)f(q,t)]. \qquad (5.24)$$

Equation (5.24) coincides with equations (5.9) and (5.11), and represents the desired differential equation for the function $f(q,t)$. This differential equation is called the generalised Fokker-Planck equation. Equation (5.24) can be written in the form of the usual continuity equation $\frac{\partial f}{\partial t}+\frac{\partial}{\partial q}J(q,t)=0$. The $f(q,t)$ can be treated as the density of the number of macrosystem copies, for which the parameter takes on the value q, and the flux – $J(q,t)=A(q)f(q,t)-\frac{1}{2}\frac{\partial}{\partial q}[B(q)f(q,t)]$, can be treated as a flux density. The second component in the last formula describes the diffusion of the macrosystem copies in space q.

The conditions (5.20)...(5.23) represent the constraints on the type of function $\varphi(q,\xi,t)$. These conditions, together with $\Delta t\to 0$; $\langle\xi\rangle^n_{\Delta t}\to 0$, $(n\geq 4)$, are satisfied only when $\varphi(q,\xi,t)$ is a Gaussian normally distributed function [41], i.e.:

$$\varphi(q,\xi,t)=\frac{1}{\sqrt{2\pi Bt}}\exp\left\{-\frac{(\xi-At)^2}{2Bt}\right\}. \qquad (5.25)$$

Equation (5.25) holds with different accuracies for a wide range of processes. Therefore, equation (5.24), as one of the mostly universal kinetic equations, is widely used for studying various processes, and allows for generalisations in the case of multiple variables. Equation (5.24) does not necessarily always allow for analytical solutions. Moreover, in the general case, this equation cannot be considered as an approximation. More specifically, continuous and discrete realisation of equation (5.24) has unambiguous correlation among the parameters of continuous and discrete processes, and increments $\Delta q = q - q_0$ have $0\{\Delta q^2\}$ infinitesimal order.

The generalised Fokker-Planck equation (5.24) is a special case of the FPK equation: $\frac{\partial f(q,t)}{\partial t} = -\frac{\partial}{\partial q}[A(q,t)f(q,t)] + \frac{1}{2}\frac{\partial^2}{\partial q^2}[B(q,t)f(q,t)]$, showing that $A(q)$ and $B(q)$ are independent of time for the processes being considered. The parameter $B(q) = \sigma^2(q)$, describing the speed of change of standard deviation in the process of evolution of q from its mean, can still be found as (constant in time) local speed of change of variance of increment of this process in the space q. This specifies the class of macrosystems, to which equation (5.24) can be applied. This class embraces macrosystems, whose motion equations include random components, changing in time much quicker than the corresponding observable parameters. In other words, the time interval, during which a value of the random component changes, has to be so small, in comparison with the time of relaxation of the examined parameter, that this random component can be considered as δ-correlated in time (e.g. white noise process). These systems being considered do not take into account any delays in the explicit forms, this is equivalent to the assumption of δ-type correlations of random components.

Hereafter we consider that the system is dissipative and basically has stationary and/or equilibrium states, which implies the possibility of inversion of $\frac{\partial f(q,t)}{\partial t}$ and/or $J(q,t)$ into zero. Practically, this possibility corresponds to the equilibrium state and/or stable motion of the dissipative system after the attenuation of all transitory processes that are influenced by the initial conditions. It is clear that the equilibrium $f_e(q)$ and stationary $f_s(q)$ PDFs of $f(q,t)$ may exist only for systems that are stable in time, and have stationary random disturbances. In these cases $A(q)$ and $B(q)$ are independent of time. But, it has to be remembered that in the general case these conditions are not sufficient.

So, the creation of the necessary theoretical prerequisites for the application of analytical methods for the investigation of the processes of change q in systems with random disturbances seems to be accomplished. However, the problem of approximating real random disturbances using white noise type processes is not trivial. The degree of conformance of the processes, described with equation (5.24), with the real evolution of the spectral parameter q for the real time series of currency exchange rates, will be tested on the available experimental data. But first, we would like to start with a statistical model estimation of this process with the generalised Fokker-Planck equation.

5.4 Statistical Model Estimation of the Evolution Processes of the Spectral Parameter

Let us apply equation (5.24) to describe of the evolution of a spectral parameter q for some currency exchange rate. Let us introduce a function $f(q,t)$, characterising the probability of different values of q in the evolution processes of a particular currency exchange rate. The function $f(q,t)$ is set in such a way that $f(q,t)dq$ defines those observations, where q belongs to the interval $(q+dq)$. We assume that $q \geq 0$. This allows the normalisation requirement for the function $f(q,t)$ to be written in the form:

$$\int_0^{+\infty} f(q,t)dq = 1, \tag{5.26}$$

and the average values of q for different time t to be found as: $\int_0^{+\infty} qf(q,t)dq = \bar{q}(t)$.

Let us examine the process of change of q with time in more detail. Firstly, the speed $\frac{dq}{dt}$ of the change of q is introduced with the Langevin equation:

$$\frac{dq}{dt} = A(q) + \tilde{w}(t), \tag{5.27}$$

where $A(q)$ is the average speed of q; $\tilde{w}(t)$ is a random component of the speed of change of q. To find the function $A(q)$ the following empirical model is also assumed:

$$\Delta\varepsilon = -C\frac{\Delta q}{q^m}, \tag{5.28}$$

where C is some constant, depending on the type of the currency being considered; $\Delta\varepsilon$ is some energy density, determining the expenses of the market participants, necessary for changing the average value of q by Δq; and m is an order parameter, characterising the types of the processes occurring. We consider $m = 1$ and obtain from equation (5.28): $\frac{\Delta q}{\Delta t} = -\frac{q\Delta\varepsilon}{C\Delta t}$. So for the average speed of change q we obtain:

$$A(q) = -kq, \tag{5.29}$$

where $k = \frac{\Delta\varepsilon}{C\Delta t}$. With equation (5.29) it is possible to simulate the behaviour of those market participants, who are interested in changes in the existing currency exchange rates, i.e. of those, who are interested in systematic declines in q. In this form, equation (5.29) is applicable only for the simulation of unstable markets with arbitrary tendencies. Thus, this form of equation (5.29) could not be accepted for the simulation of evolution processes of the main world currencies. The terms market organisation and stability imply the presence of at least two parties in the market. This allows equation (5.29) to be written as:

$$A(q) = -k(q - q_e), \tag{5.30}$$

where q_e represents some equilibrium value, suitable for all market participants. The concept of an equilibrium value imposes the restriction $k \geq 0$, also implying the stability of the processes and the limited abilities of all parties involved; otherwise the simulated system becomes unstable. If the system is unstable ($k < 0$), this implies the inability of the market participants for trade-off, what is equivalent to either elimination of one of the parties involved, or to their separate operations. It is assumed that for $k \geq 0$ equation (5.30) is suitable: for example, to the simulation of self-organising stable markets without central planning. For most of the world's main currencies, equation (5.30) is not fully applicable, since it does not take into account the presence of the key participants at foreign exchange markets, which are the central banks (CBs) (or the control centres) of countries, participating in currency exchange operations, which possess a nearly unlimited ability to compare with other FX-market participants.

Let us assume that the actions of the CBs result in a stabilisation of the market processes, and also affect the intensity of stabilisation, which substantially increases when q declines. Then, instead of equation (5.30), we will use:

$$A(q) = -k(q - \frac{q_{ec}^2}{q}), \tag{5.31}$$

where activities of the CBs in the space of change in q are considered via $\frac{q_{ec}^2}{q}$, in which $q_{ec}^2 = \frac{cq_0^2}{k}$ characterises some current equilibrium value q_{ec}, acceptable for the CBs, and cq_0^2 define the current activities of the CBs for keeping this equilibrium value of q_{ec}. For $k \geq 0$, equation (5.31) can be used for the simulation of the performance of those stable markets, where central planning opposes arbitrage. It is assumed that the impact of any part, interested in keeping the existing tendencies, is either negligible or absolutely coincides with the actions performed by the central planning control centres. This situation is rather typical of the processes with memory, which do not have pronounced trends. Substituting equation (5.27) with equation (5.31) we get: $\frac{dq}{dt} = -k(q - \frac{q_{ec}^2}{q}) + \widetilde{w}(t)$. Assuming that the fluctuating component $\widetilde{w}(t)$ is δ-correlated in time, and $B(q) = \sigma^2(q) = const = 2b$, then using equation (5.24) for the function $f(q,t)$ it is possible to write:

$$\frac{\partial f}{\partial t} = k\frac{\partial}{\partial q}\left[(q - \frac{q_{ec}^2}{q})f\right] + b\frac{\partial^2 f}{\partial q^2}. \tag{5.32}$$

To solve equation (5.32), we use the function $f_0(q)$ as the initial condition, which defines the initial distribution q and satisfies the normalisation requirement: $\int_0^{+\infty} f_0(q)dq = 1$. Since the function $f(q,t)$ has to satisfy the condition (5.26) at any time t, it is possible to assume that this function $f(q,t)$ is subject to the interval $(0,+\infty)$, non-singular for $q = 0$, and satisfies the condition $f(q,t) \to 0$ for $q \to +\infty$.

Let us use the variable separation method to represent the function $f(q,t)$ in the form $f(q,t) = R(q)T(t)$. This allows equation (5.32) to be represented in the form of the following simultaneous equations:

$$\begin{cases} \dfrac{dT}{dq} = \lambda T \\ \dfrac{d}{dq}\left[k(q - \dfrac{q_{ec}^2}{q})R + b\dfrac{dR}{dq} \right] = \lambda R \end{cases} \tag{5.33}$$

Let us introduce the new variable $x = \dfrac{kq^2}{2b}$ into the second equation of system (5.33), and make the substitution: $R(q) = x^{\Theta/2}\exp(-x)\Psi(x)$, where $\Theta = \dfrac{kq_{ec}^2}{b}$. After some simple but cumbersome transformations, it is possible to obtain the following equation for the function $\Psi(x)$:

$$x\frac{d^2\Psi}{dx^2} + \left[\frac{(\Theta+1)}{2} - x\right]\frac{d\Psi}{dx} - \left(\frac{\lambda}{2k}\right)\Psi = 0. \tag{5.34}$$

Following [162], equation (5.34) has a solution, subject to infinity and non-singular in zero, when $-\dfrac{\lambda}{2k}$ is an integer. Then, the generalised Laguerre polynomials in the form $L_i^{\left(\frac{\Theta-1}{2}\right)}(x)$ represent the solution to equation (5.34) for characteristic values $\lambda = -2ki$, where $i = 0,1,2,\ldots$. These polynomials can be obtained with the generalised Rodriguez equation:

$$L_i^{\left(\frac{\Theta-1}{2}\right)}(x) = e^x x^{-\left(\frac{\Theta-1}{2}\right)} \frac{d^i}{dq^i}\left(e^{-x} x^{i+\left(\frac{\Theta-1}{2}\right)} \right). \tag{5.35}$$

In particular, from equation (5.35): $L_0^{\left(\frac{\Theta-1}{2}\right)} = 1$; $L_1^{\left(\frac{\Theta-1}{2}\right)} = 1 - x + \dfrac{\Theta-1}{2}$, etc. Then, taking into account the initial variables for the function $f(q,t)$, and considering equation (5.33), it can be concluded that functions of the type: $\left\{ q^{\Theta}\exp\left(-\dfrac{kq^2}{2b}\right)L_i^{\left(\frac{\Theta-1}{2}\right)}\left(\dfrac{kq^2}{2b}\right)\exp(-2kit) \right\}$, $i = 0,1,2,\ldots$, are partial solutions to the initial equation (5.32). Consequently, the general solution to this equation can be represented as:

$$f(q,t) = q^{\Theta} \exp\left(-\frac{kq^2}{2b}\right) \sum_{i=0}^{+\infty} C_i L_i^{\left(\frac{\Theta-1}{2}\right)}\left(\frac{kq^2}{2b}\right) \exp(-2kit), \tag{5.36}$$

where C_i are constants which can be obtained from the initial and normalisation conditions of the function $f(q,t)$. Since for $\frac{\Theta-1}{2} > -1$, ($\Theta > -1$):

$$\int_0^{+\infty} e^{-x} x^{\left(\frac{\Theta-1}{2}\right)} \left[L_i^{\left(\frac{\Theta-1}{2}\right)}\right]^2 dx = i!\Gamma\left(i + \frac{\Theta-1}{2} + 1\right), \tag{5.37}$$

then transforming both sides of equation (5.36) with $\int_0^{+\infty} L_i^{\left(\frac{\Theta-1}{2}\right)}\left(\frac{kq^2}{2b}\right) dq$, and taking into account equation (5.37) and the initial conditions, the following equation can be obtained:

$$C_i = \frac{\Theta^{\left(\frac{\Theta+1}{2}\right)} \int_0^{+\infty} f_0(q) L_i^{\left(\frac{\Theta-1}{2}\right)}\left(\frac{kq^2}{2b}\right) dq}{q_{ec}^{(\Theta+1)} 2^{\left(\frac{\Theta-1}{2}\right)} \Gamma\left(i + \frac{\Theta+1}{2}\right) i!}. \tag{5.38}$$

Substituting equation (5.36) with (5.38) we get the required equation for the simulated distribution function $f(q,t)$. The obtained distribution $f(q,t)$ is not exactly normal, but could be approximated to one. Indeed, being a single-mode mesokurtic distribution for any $t > 0$, the function $f(q,t)$ behaves exponentially $\sim \exp\left(-\frac{kq^2}{2b}\right)$, with an order parameter $m_1 = m + 1 = 2$, to the right of the mode; and behaves as a power function $\sim q^{\Theta}$, with an order parameter Θ, to the left of the mode. Parameters *k*, *b* and q_{ec} can be considered as external, since their values are determined by characteristics of a particular currency and by the conditions of the evolution processes. For instance, the parameter *k* is determined mainly by the "physical costs" of the market participants, aimed at changing *q*; the parameter q_{ec} depends on the "efforts" and the estimates of the CBs of the participating countries; parameter *b* can be considered as a "complex" characteristic, determining the degree of impact of any incoming information.

Figure 5.1 illustrates changes in the PDF $f(q,t)$, determined by equations (5.36) and (5.38) for some values of *k*, q_{ec} and $\Theta = \frac{cq_0^2}{b}$ at different moments of time $t \geq t_0$. These results indicate an important characteristic of the evolution of the spectral parameter *q* that is the widening of the plot of distribution function $f(q,t)$ at the initial stage of its evolution. The width of the graph of the distribution function $f(q,t)$ indirectly characterises volatility, which grows at the initial stage of the evolution process of the parameter *q* despite the fact that the value of *b* stays constant. As opposed to the impact of volatility of prices, the behaviour of the system in the space q has its own specificity. When volatility is small, prices usually trend towards longer growths or drops, but, if volatility is high, growths or drops in prices slow down, tending to revert the current process.

$q_0 = 1.5,\ q_{ec} = 1.0$ $\qquad\qquad$ $q_0 = 1.0,\ q_{ec} = 1.5$

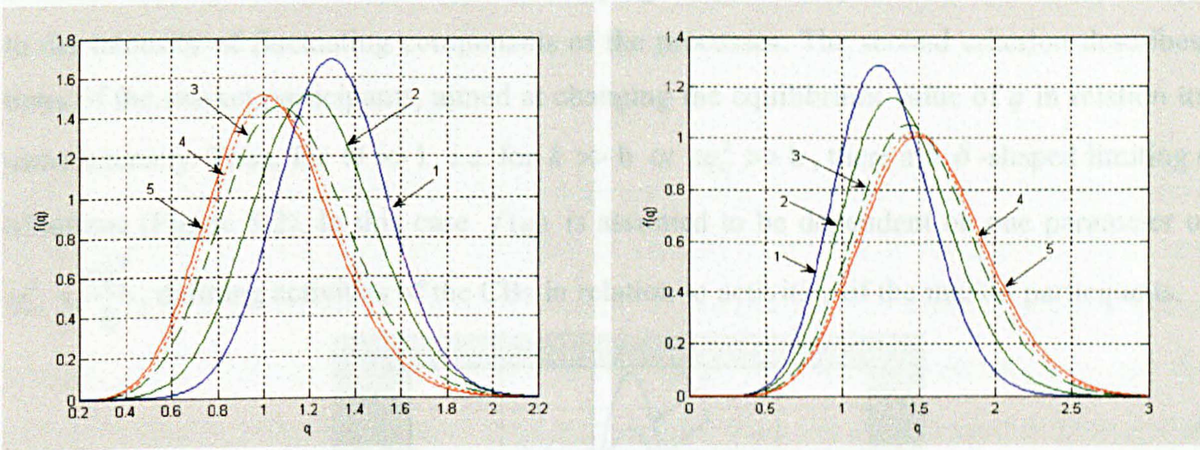


Figure 5.1. Evolution of the PDF $f(q,t)$ under Θ = 7 for Initial Conditions q_0 and q_{ec}

1: $kt = 0.3$; **2:** $kt = 0.5$; **3:** $kt = 1.0$; **4:** $kt = 1.5$; **5:** $kt = 3.0 - \infty$

At the same time according to the proposed model (Figure 5.1) all tendencies of changes in $q(t)$ in the space q remain and are accompanied by a widening of the plot of the distribution function $f(q,t)$. This effect has to be experimentally tested as it could be of a great importance for analysing, forecasting and identification of real data.

The limiting distribution $f(q)$ can be analytically obtained from equation (5.36) by proceeding to the limit under $t \to \infty$, and taking into account equation (5.38), in the form of:

$$f(q) = C_0 q^{\Theta} \exp\left(-\frac{kq^2}{2b}\right). \tag{5.39}$$

Here: $C_0 = \dfrac{2}{\Gamma\left(\frac{\Theta+1}{2}\right)}\left(\dfrac{\Theta}{2q_{ec}^2}\right)^{\left(\frac{\Theta+1}{2}\right)}$, where $\Gamma\left(\frac{\Theta+1}{2}\right)$ is the gamma function. (5.40)

The dependence of the limiting distribution $f(q) = \lim_{t\to+\infty} f(q,t)$ from the parameter Θ is presented on Figure 5.2. From equations (5.39) and (5.40) it follows that during the evolution process in time, under a stable (unchanging) external environment, some limiting distribution $f(q)$, independent of the initial distribution $f_0(q)$, can be obtained. Distribution $f(q)$ corresponds to the dynamic equilibrium conditions of the system, determined with $J(q,t) = 0$. Based on equations (5.39) and (5.40), it is possible to draw a number of qualitative conclusions, related to the control over the occurring processes. Since $\dfrac{k}{b} = \dfrac{\Theta}{q_{ec}^2}$, the limiting distribution

$f(q)$ is independent of all parameters k, b and q_{ec} in an explicit form, but is dependent on the two combinations, or criteria based on these parameters: $\Theta = \frac{kq_{ec}^2}{b} = \frac{cq_0^2}{b}$ and k/b. The first criterion describes actions of the CBs for keeping the present equilibrium value of q in relation to the intensity of fluctuating components of the processes. The second criterion describes actions of the market participants, aimed at changing the equilibrium value of q in relation to the same intensity. Then, for $\Theta >> 1$, i.e. for $k >> \mathrm{b}$ or $cq_0^2 >> \mathrm{b}$, there are δ-shaped limiting distributions (Figure 5.2). In this case $f(q)$ is assumed to be dependent on one parameter only: $q_{ec}^2 = \frac{cq_0^2}{k}$, defining activities of the CBs in relation to activities of the market participants.

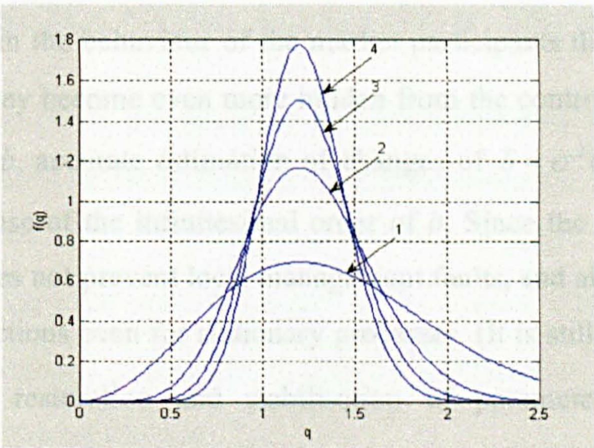


Figure 5.2. Dependence of limited PDF $f(q)$ from the Parameter Θ ($q_{ec} = 1.2$)
1: $\Theta = 2$; **2:** $\Theta = 6$; **3:** $\Theta = 10$; **4:** $\Theta = 14$

So in a number of cases ($\Theta >> 1$) the limiting distribution is determined with only one parameter – q_{ec}, and increases in k result in only minor changes of the distribution $f(q)$ (Figure 5.2). In other words, under effective central planning, explicitly arbitrage behaviour of the market participants ($k >> \mathrm{b}$, and values of k/b are high) is controlled by the CBs, and for long time intervals in most cases these have no serious affects on the general situation of the market ($q \approx q_{ec}$). This implies that the occurrence of any "anomalies" in the market is totally determined by these (anomalous) external circumstances, or results from the ineffective actions of the CBs. The higher the intensity of arbitrage (k increases), the more chances the CBs have to minimise possible negative after-effects ($f_0(q) \to f(q)$ is quicker), because in this case from equation (5.36) with increasing time t, parameters $\exp(-2kit)$, for any $i \neq 0$, tend to zero quickly. For instance, if we assume that $q \approx q_{ec}$ with a 5% level of significance, then it is possible to consider that this condition already holds for $kt \approx 3$, for any $i \neq 0$ (Figure 5.1), i.e. the damping time of all of the transition processes can be estimated for $t_s \approx 3/k$. If we consider that under a standard environment for daily changes $k \approx 1$, and under some anomalies it increases to $k \approx 3$, then it becomes clear that the time of market stabilisation – t_s – amounts correspondingly to three and one days. This circumstance imposes restrictions on the structure of the time series, used for realising the processes of management. If in the first case non-stationary distributions $f(q,t)$ could be examined with daily data, then in the second case the analysis of intraday data is required.

Another important conclusion that can be drawn from the above discussion is related to the type of management process. This type is related to changes (increases or declines) of the parameter $b = \sigma^2(q)/2$, which is the analogue of volatility in the space q. For instance, when b increases, tendencies for arbitrage in the behaviour of the market participants diminish (k/b and Θ decline). In other words, they become even more hidden from the control centres. Independent of increases or declines in b, accurate estimation of changes of $b = \sigma^2(q)/2$ (for $\sigma(q) << 1$) is rather complicated because of the infinitesimal order of b. Since the system stays in dynamic equilibrium only, this does not prevent local management faults, and also diminishes the effectiveness of managerial actions even for stationary processes. (It is still difficult to find managerial actions, aimed at restoration and stabilisation of parameters $\Theta = \frac{kq_{ec}^2}{b} = \frac{cq_0^2}{b}$ and $q_{ec}^2 = \frac{cq_0^2}{k}$.) As a result, the probability of the occurrence of local situations when $q(t) > q_{ec}$; $q(t) > 1$ or $q(t) < q_{ec}$; $q(t) < 1$ increases significantly. For instance, with increasing b, Θ declines, and plots of $f(q,t)$ widen (Figure 5.2). The behaviour of the system in the space q, resulting from changes in b, has its own specificity in comparison with the nature of the impact of the volatility of the price characteristics. For minimal or for declining values of b (k/b and Θ increase), the plots of $f(q,t)$ shrink (Figure 5.2), and the determining role is played by the external conditions and by local management faults, caused mainly by errors in estimates of cq_0^2, and consequently of Θ and q_{ec}.

The situations considered above, lead either to the retention or development (acceleration) of the existing tendencies in the market, with the possibility of leaps ($q(t) > 1$), or to slow-downs or changes of the existing tendencies with the increasing alternation of prices ($q(t) < 1$). Therefore, this theoretically obtained result has to be experimentally tested. Similarly, consequences arise from local changes in k and q_{ec}, and/or from their local estimation errors. The decline in k is accompanied by an increase in the time of relaxation of the whole system – t_s, and the estimation errors of rising q_{ec}, for $q_{ec}>1$, are more significant because of the non-linearity in the ratio $\Theta = \frac{kq_{ec}^2}{b} = \frac{cq_0^2}{b}$. Using explicit forms of equations (5.36) and (5.39), it is possible to find solutions to many practical problems. Taking into account the qualitative nature of the obtained results, at the first stage of this research, we only use stationary distributions $f(q)$, which virtually implies the use of asymptotic, i.e. stationary statistical methods of analysing dissipative systems. Leaving cumbersome analytical argument aside, this approach allows stationary distributions $f(q)$ to be considered in an explicit form, even for evidently more complex systems.

5.5 The Analysis of the Evolution Process of the Spectral Parameter Using Asymptotic Methods

From equations (5.39) and (5.40) it is possible to conclude that during the evolution process of currency exchange rates under a stable external environment, there is some limiting distribution $f(q)$, which is independent of time and the initial distribution $f_0(q)$. This limiting distribution $f_j(q)$ corresponds to some local stationary state j of the system, which is determined with its dynamic equilibrium conditions in the form of a continuity equation under $J(q,t)=0$. In this case equation (5.24) can be simplified to:

$$\frac{d}{2dq}\left[B(q)f_j(q)\right] - A(q)f_j(q) = 0, \tag{5.41}$$

and any stationary state j of the system can be defined with the limiting distribution $f_j(q)$. It follows from equation (5.41) that any change in the stationary state j of the system being considered, and its transition to the stationary state $j+1$, is possible only after a change in the external parameters of the system. The evolution process of the system can be presented as a set of the series of stationary states, which the dissipative system passes through in an interaction with the external environment. Under some conditions, the external parameters of the system depend only on the difference between t_{j+1} and t_j, rather than on the values of t_{j+1} and t_j. In turn, changes in the external parameters of the dissipative system result in corresponding changes of a limiting distribution $f(q)$, which causes the transfer from an initial limiting distribution $f_j(q)$ to the new limiting distribution $f_{j+1}(q)$. The evolution process of the spectral parameter q can be represented as a set of consecutive parameters $q=q(t)$ and/or $q=q(\Delta t)$, determined by the conditions of the interrelation of the system with the external environment. If a constant time interval between registration of the last two consecutive states of the system exceeds the time for the end of all transitory processes occurring in the system, then a set of stationary states of this system will be generally observed. So the continuous evolution processes of the spectral parameter $q = q(t)$ and/or $q = q(\Delta t)$, can be approximately presented with a consecutive discreet chain of limiting distributions $f_j(q)$, depending only upon external parameters of the system, typical of the previous registration moment t_j or the difference Δt.

Let us now analyse the behaviour of the limiting distributions $f_j(q)$ on the examples. For $A(q)$ satisfying equation (5.29), and taking into account all assumptions made, for $B(q) = \sigma^2(q) = const = 2b$ from equation (5.41) it is possible to get:

$$f_{j+1}(q) = C_j \exp\left(-\frac{k_j q^2}{2b_j}\right) = \sqrt{\frac{2k_j}{\pi b_j}} \exp\left(-\frac{k_j q^2}{2b_j}\right). \tag{5.42}$$

Due to the imposed constraint $q \geq 0$, the stationary distributions (5.42) (Figure 5.3) are truncated (by half from the left) normal distributions with mode $q_m = 0$ and mean:

$$\overline{q}_{j+1} = \sqrt{\frac{2k_j}{\pi b_j}} \int_0^{+\infty} q \exp\left(-\frac{k_j q^2}{2b_j}\right) dq = \sqrt{\frac{2b_j}{\pi k_j}} \neq 0.$$

The kind of distribution (5.42) is determined by the combination b_j / k_j of the two external parameters k_j, b_j, and can simulate the behaviour of the system under unstable markets where speculative (arbitrage) tendencies dominate. It is evident, that on average these markets are relatively (conditionally) stable. The force of the stabilisation process increases either with a decline in speculative expectations, or as a result of an increase in the intensity of random impacts, i.e. when b_j / k_j grows.

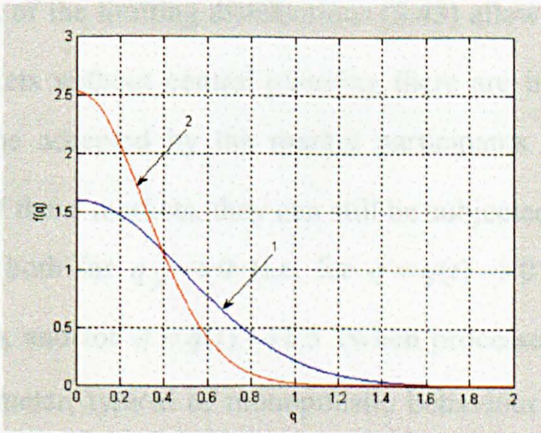


Figure 5.3. Dependence of the PDF (5.42) from *b/k* ($q_e = 0$)

1: *b/k* = 0.25; **2:** *b/k* = 0.1

Let us turn to the analysis of the behaviour of self-organising, stable markets without central planning. For this, keeping all assumptions unchanged, consider equation (5.30). Then, for $B(q) = \sigma^2(q) = 2b = const$ from equation (5.41) it is possible to get:

$$f_{j+1}(q) = C_j \exp\left(-\frac{k_j (q - q_{je})^2}{2b_j}\right). \tag{5.43}$$

Stationary distributions (5.43) (Figures 5.4) are truncated normal distributions with left-truncated variables, depending on the three external parameters k_j, b_j and q_{je}, with mode $q_{jm} = q_{je}$ and mean: $\overline{q}_{j+1} = C_j \int_0^{+\infty} q \exp\left(-\frac{k_j (q - q_e)^2}{2b_j}\right) dq$. Together with a variable truncating parameter, the kind of distributions (5.43) (in comparison with (5.42)), is determined by the two possible combinations of the three external parameters k_j, b_j and q_{je}, i.e. b_j / k_j and q_{je}.

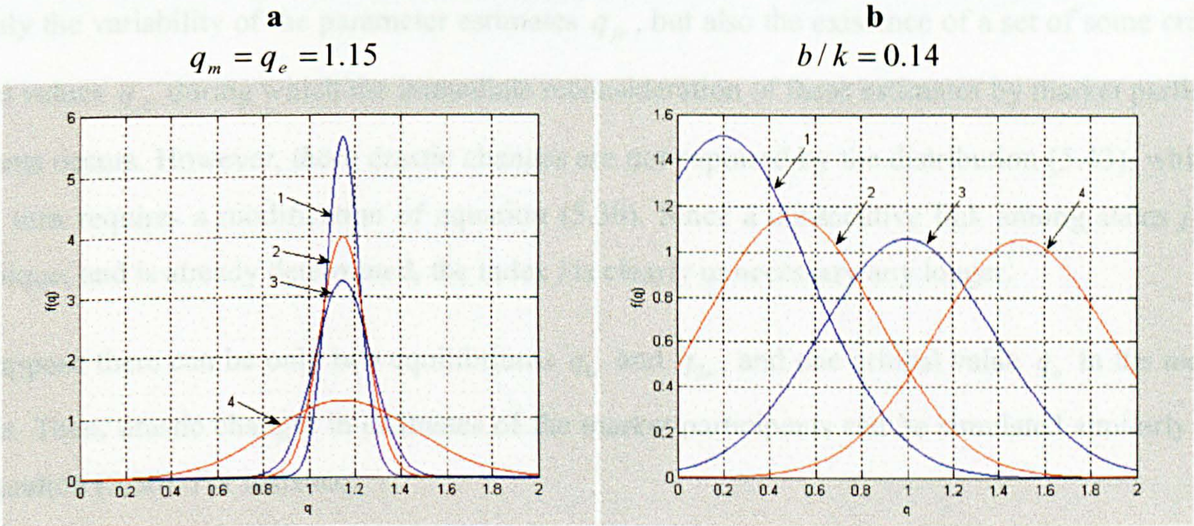


Figure 5.4. Dependence of the PDF (5.43) from *b/k* and $q_m = q_e$

a - 1: *b/k* = 0.005; **2 :** *b/k* = 0.01; **3:** *b/k* = 0.015; **4:** *b/k* = 0.1
b - 1 : $q_m = q_e = 0.2$; **2 :** $q_m = q_e = 0.5$; **3 :** $q_m = q_e = 1.0$; **4:** $q_m = q_e = 1.5$

Distributions (5.42) can be considered as a special case of distribution (5.43), half-truncated for $q_{jm} = q_{je} = 0$. The analysis of the limiting distributions (5.43) allow for the conclusion that for self-organising stable markets without central planning there are infinitely many equilibrium values q_{je}, which could be accepted by the market participants. As a result, despite self-organisation and stability of these markets, they can still be subjected to arbitrage. Indeed, arbitrage is explicitly possible both for $q_{je} \to 0$ (i.e. for $q = q(t) \to 0$, when processes of white noise dominate the system), and for $q = q(t) \to 1.5$ (when processes of black noise with high values of the spectral parameter, typical of monopolistic behaviour of the market participants ($q_{je} \to 1.5$) are observed). In other words, the notion of a self-organising stable market is not equivalent to the notion of ECM.

We have the opportunity to describe the structure of processes occurring in the market at a particular moment of time with parameter $q = q(t)$. In fact, this implies that under self-organising stable markets, the parameter $q = q(t)$ can be considered as a complex macroeconomic parameter, characterising a markets' dynamic structure at different points of time. This type of market can be considered as ECM only roughly, when $q = q(t) = 1$. The degree of this approximation can be estimated by the difference $\Delta q_q = q_{je} - \bar{q}_j$, which also can be considered as a measure of the approximation of the distribution $f_j(q)$ to a normal distribution. Thus, the introduction and implementation of the complex parameter $\Delta q_q / q$ is justified, for identification problems at least. The existence, for a considered type of market, of a set of equilibrium values, implies not only the variability of the parameter estimates q_{je}, but also the existence of a set of some critical values q_{jc} during which the immediate reconsideration of these estimates by market participants occurs. However, these drastic changes are not captured by the distribution (5.43), which in turn requires a modification of equation (5.30). Since a consecutive link among states j is unique, and is already determined, the index j is clearly unnecessary any longer.

Suppose there can be only two equilibriums q_{1e} and q_{2e}, and one critical value q_c in the market. Then, drastic changes in estimates of the market participants can be simulated similarly to equation (5.30). For instance:

for $q \le q_c$: $A(q) = -k_1(q - q_{1e})$, and

for $q \ge q_c$: $A(q) = -k_2(q - q_{2e})$, or

$$\text{for } q \ne q_c: \; A(q) = -\left(k \pm k_a \frac{|q - q_c|}{q - q_c}\right)\left(q - q_e \pm q_a \frac{|q - q_c|}{q - q_c}\right), \tag{5.44}$$

where k_1, k_2 and q_{1e}, q_{2e} equal correspondingly to $k \pm k_a \frac{|q - q_c|}{q - q_c} > 0$ and $q_e \pm q_a \frac{|q - q_c|}{q - q_c} > 0$, and q_e, q_a and k, k_a are some constants. Having determined b_1 and b_2 in the same way, it is possible to write:

$$b_{1,2} = b_e \pm b_a \frac{|q - q_c|}{q - q_c} > 0. \tag{5.45}$$

Taking into account equations (5.44) and (5.45), for $q_{1e} < q_c < q_{2e}$, $q \ne q_c$, from equation (5.41) it is possible to get distributions:

$$f(q) = C\exp\left(-\left(k \pm k_a \frac{|q - q_c|}{q - q_c}\right)\frac{\left(q - q_e \pm q_a \frac{|q - q_c|}{q - q_c}\right)^2}{2\left(b_e \pm b_a \frac{|q - q_c|}{q - q_c}\right)}\right), \tag{5.46}$$

which for $q_{1e} < q_c < q_{2e}$, $k_1 = k_2 = k$ and $b_1 = b_2 = b$ can be modified into:

$$f(q) = C_0 \exp\left(\frac{-k\left(q - q_e - q_a \dfrac{|q - q_c|}{q - q_c} \right)^2}{2b} \right), \tag{5.47}$$

where C_0 is a constant, determined from the normalisation condition.

Distributions (5.46) and (5.47) belong to the class of bi-modal distributions with modes $q_{1m} = q_{1e}$ and $q_{2m} = q_{2e}$, which have isometric values $f(q_{1m})$ and $f(q_{2m})$ (Figure 5.5).

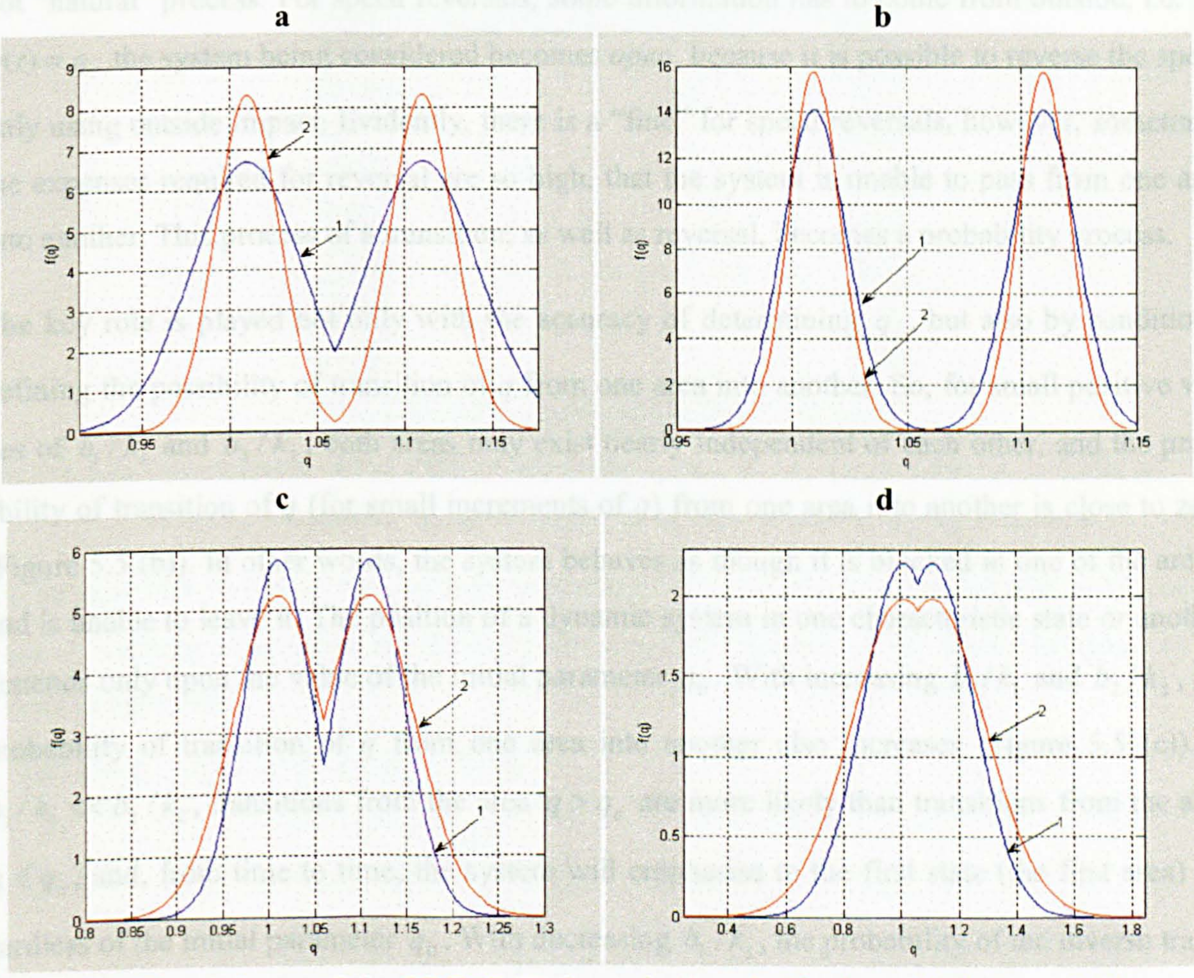


Figure 5.5. Dependence of the PDF (5.47) from $b_1 / k_1 = b_2 / k_2$ **(**$q_c = 1.06$**,** $q_{1e} = 1.01$**,** $q_{2e} = 1.11$**)**

a - $b_1 / k_1 = b_2 / k_2$ **1:** 0.001, **2:** 0.00033; **b** - $b_1 / k_1 = b_2 / k_2$ **1:** 0.0002, **2:** 0.000125;

c - $b_1 / k_1 = b_2 / k_2$ **1:** 0.0014, **2:** 0.0025; **d** - $b_1 / k_1 = b_2 / k_2$ **1:** 0.02, **2:** 0.033

The distributions (5.46) and (5.47), together with variable left-truncation, are now determined with k_1 and k_2, b_1 and b_2, q_{1e} and q_{2e}, and with q_c, i.e. with b_1 / k_1, q_{1e} and b_2 / k_2, q_{2e}, and with parameter q_c. Virtually, we obtain two areas where q could change: the first for $q(t) \approx q_{1e}$, and the second for $q(t) \approx q_{2e}$. If we assume, that the first area exists for $q_{1e} \approx 1$ (the type of processes close to brown noise), and the second area exists for $1 < q_{1e} \leq 1.5$, then each area can be considered as some characteristic state, in which the analysed system is in.

What does q_c actually characterise in the proposed model? To answer this question, it is worth noting that at time t, when $q(t) = q_c$, speed reverses and a qualitatively new situation emerges. For $q(t) = q_c$, the average speed of the process takes on two different values, and the function dq/dt breaks down. So, two areas emerge, in which the system "naturally" evolves according to the corresponding laws. It has to be taken into account, that for a closed system a reversal is a not "natural" process. For speed reversals, some information has to come from outside, i.e. for $q(t) = q_c$ the system being considered becomes *open*, because it is possible to reverse the speed only using outside impact. Evidently, there is a "fine" for speed reversals, however, sometimes the expenses required for reversal are so high, that the system is unable to pass from one area into another. This process of a transition, as well as reversal, becomes a probability process.

The key role is played not only with the accuracy of determining q_c, but also by conditions, defining the possibility of transition of q from one area into another. So, for small positive values of b_1/k_1 and b_2/k_2, both areas may exist nearly independent of each other, and the probability of transition of q (for small increments of q) from one area into another is close to zero (Figure 5.5 (b)). In other words, the system behaves as though it is blocked in one of the areas, and is unable to leave it. The position of a dynamic system in one characteristic state or another depends only upon the value of the initial parameter q_0. With increasing b_1/k_1 and b_2/k_2, the probability of transition of q from one area into another also increases (Figure 5.5 (c)). If $b_1/k_1 << b_2/k_2$, transitions from the area $q > q_c$ are more likely than transitions from the area $q < q_c$, and, from time to time, the system will emphasise to the first state (the first area) regardless of the initial parameter q_0. With decreasing b_1/k_1, the probability of the inverse transfer goes down. The relative time of the presence of the system in one of the areas (states) depends not only upon b_1/k_1 and b_2/k_2, but also upon q_c (because the area under the distribution curve depends on q_c). Otherwise ($b_1/k_1 >> b_2/k_2$), transitions from the area $q < q_c$ are more likely, and while leaving it, from time to time, the system will emphasise the second state (the second area) regardless of the initial parameter q_0 (Figure 5.6).

When $b_1/k_1 \approx b_2/k_2$, the probabilities of transition from one area into another become comparable, i.e. the system periodically passes from one state (area) to the other. The relative time of presence of the system in each of the areas (states) depends only upon q_c. With increasing b_1/k_1 and b_2/k_2, the distribution could be roughly or falsely classified as a single-mode mesokurtic distribution (Figure 5.5 (d)). The above analysis is applicable to modeling dynamic systems with only one critical value q_c, i.e. for systems with only two characteristic states. In a real dynamic system there could be many more critical values, which in turn requires corresponding modifications to the equations presented above. Regardless of this modification, it can be claimed that the accurate determination and practical application of the parameters q_c is of high practical importance, at least with regards to the problems of identification and forecasting.

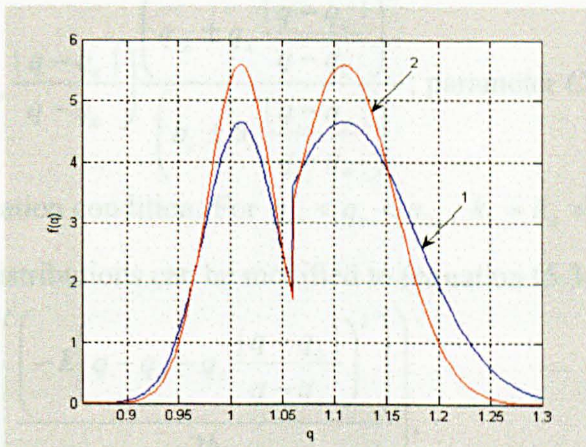


Figure 5.6. Dependence of the PDF (5.46) from b_1/k_1, b_2/k_2 ($q_c = 1.06$, $q_{1e} = 1.01$, $q_{2e} = 1.11$)
1: $b_1/k_1 = 0.00016$, $b_2/k_2 = 0.002$; **2:** $b_1/k_1 = 0.00033$, $b_2/k_2 = 0.0014$

Let us now turn to considering of the behaviour of self-organising stable markets with central control. Keeping all previous assumptions, let us modify equation (5.31) similar to equations (5.44) in the form:

for $q \le q_c$: $A(q) = -k_1\left(q - q_{1e} - \frac{q_{1ec}^2}{q}\right)$,

for $q \ge q_c$: $A(q) = -k_2\left(q - q_{2e} - \frac{q_{2ec}^2}{q}\right)$, or

$$\text{for } q \ne q_c\text{: } A(q) = -\left(k \pm k_a \frac{|q-q_c|}{q-q_c}\right)\left(q - q_e \pm q_a \frac{|q-q_c|}{q-q_c} - \frac{\left(q_{ec} \pm q_s \frac{|q-q_c|}{q-q_c}\right)^2}{q}\right), \quad (5.48)$$

where k_1, k_2; q_{1e}, q_{2e} and q_{1ec}, q_{2ec} equal correspondingly to $k \pm k_a \frac{|q-q_c|}{q-q_c} > 0$; $q_e \pm q_a \frac{|q-q_c|}{q-q_c} > 0$, $q_{ec} \pm q_s \frac{|q-q_c|}{q-q_c} > 0$, and q_e, q_a, q_s, q_{ec} and k, k_a are some constants.

Having determined b_1 and b_2 similar to equation (5.45), it is possible to write:

$$b_{1,2} = b_e \pm b_u \frac{|q - q_c|}{q - q_c} > 0. \tag{5.49}$$

Then, from equation (5.41), for $q_{1e} < q_c < q_{2e}$, $q \neq q_c$, and with regards to equations (5.48) and (5.49), it is possible to get the following distributions:

$$f(q) = Cq^{\Theta} \exp\left(-\left(k \pm k_s \frac{|q - q_c|}{q - q_c} \right) \frac{\left(q - q_e \pm q_s \frac{|q - q_c|}{q - q_c} \right)^2}{2\left(b_e \pm b_s \frac{|q - q_c|}{q - q_c} \right)} \right), \tag{5.50}$$

where $\Theta_{1,2} = \Theta = \left(k \pm k_s \frac{|q - q_c|}{q - q_c} \right) \frac{\left(q_{ec} \pm q_s \frac{|q - q_c|}{q - q_c} \right)^2}{\left(b_e \pm b_s \frac{|q - q_c|}{q - q_c} \right)}$; parameter C is a constant, and determined from the normalisation condition. For $q_{1e} < q_c < q_{2e}$, $k_1 = k_2 = k$, $b_1 = b_2 = b$ and $q_{1ec} = q_{2ec} = q_{ec}$, these distributions can be modified to (equation (5.39)):

$$f(q) = C_0 q^{\Theta} \exp\left(\frac{\left(-k\left(q - q_e - q_s \frac{|q - q_c|}{q - q_c} \right)^2 \right)}{2b} \right), \tag{5.51}$$

where $\Theta = \frac{kq_{ec}^2}{b}$; and C_0 is a constant, determined from the normalisation condition.

In general, distributions (5.50) and (5.51) also belong to the class of bi-modal distributions with modes $q_{1m} = q_{1e}$ and $q_{2m} = q_{2e}$. Unlike distributions (5.46), (5.47), due to the presence of parameter q^{Θ}, distributions (5.50) and (5.51) have different values: $f(q_{1m})$, $f(q_{2m})$ (Figure 5.7). Despite the existence of eight external parameters: k_1 and k_2; b_1 and b_2; q_{1e} and q_{2e}; q_{1ec} and q_{2ec}, and also of parameter q_c, the kind of distributions (5.50) and (5.51), together with variable left-truncation, is determined with just two combinations of the eight external parameters, i.e. with parameters $\Theta_1 = \frac{k_1 q_{1ec}^2}{b_1}$, q_{1e} and $\Theta_2 = \frac{k_2 q_{2ec}^2}{b_2}$, q_{2e}, and with parameter q_c. Parameter q_c requires further description here. For $q(t) = q_c$, some market participants, influenced by external information, are able to change their present positions, either supporting activities of the control central, or opposing them. In other words, for $q(t) = q_c$ some innovation process, able to change the behaviour of the system dramatically, is possible. Similar to the distributions

(5.46) and (5.47), we still have two areas where q could change: the first for $q(t) \approx q_{1e}$, and the second for $q(t) \approx q_{2e}$. Each of these areas can still be considered as some characteristic state, in which the system is in. If in relation to b_1 / k_1 and b_2 / k_2 all assumptions made while analysing distributions (5.46) and (5.47) maintain, but the impact of parameters q_{1ec} and q_{2ec} on the distributions (5.50) and (5.51) requires further consideration.

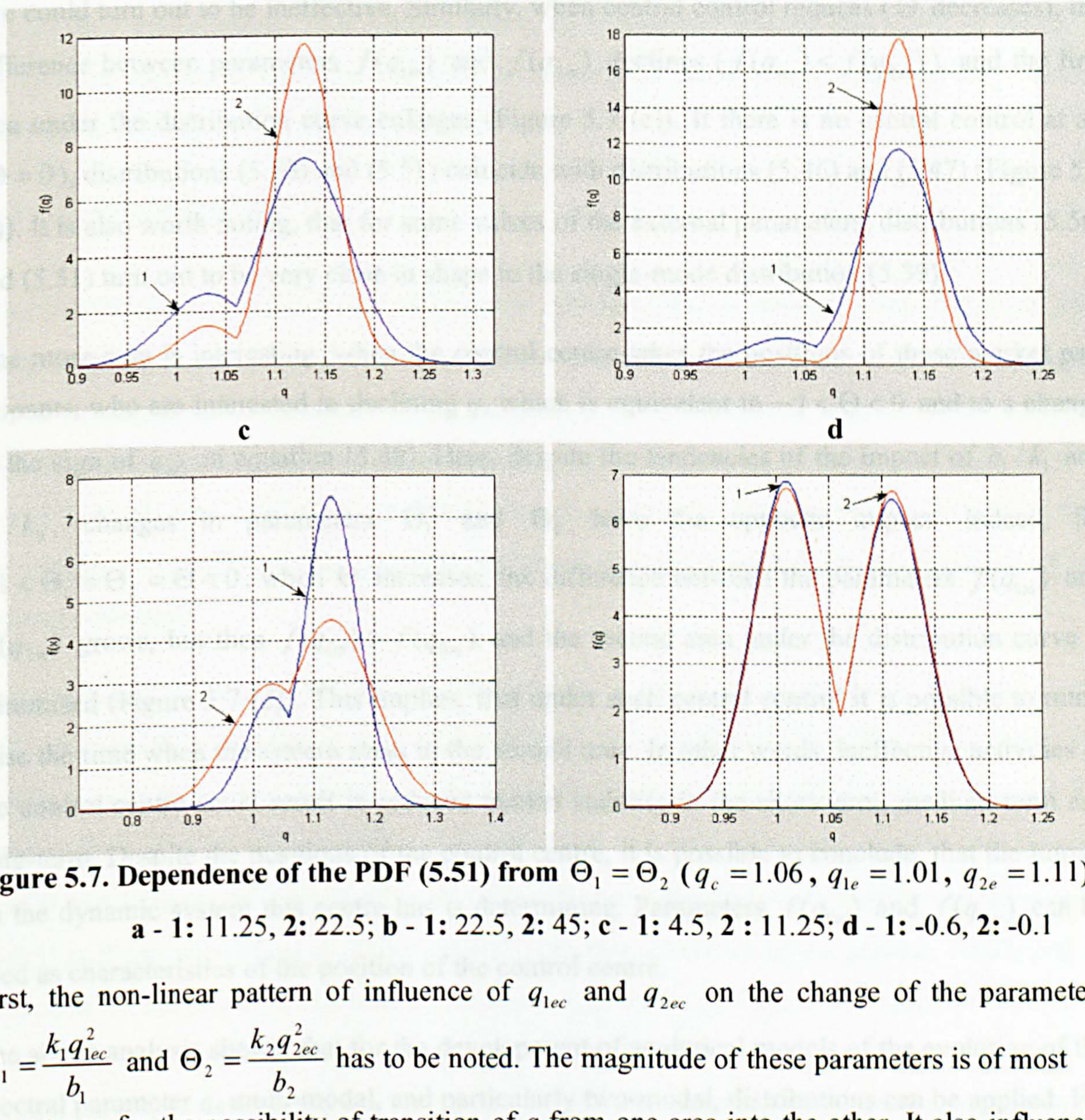


Figure 5.7. Dependence of the PDF (5.51) from $\Theta_1 = \Theta_2$ ($q_c = 1.06$, $q_{1e} = 1.01$, $q_{2e} = 1.11$)

a - 1: 11.25, **2:** 22.5; **b - 1:** 22.5, **2:** 45; **c - 1:** 4.5, **2 :** 11.25; **d - 1:** -0.6, **2:** -0.1

First, the non-linear pattern of influence of q_{1ec} and q_{2ec} on the change of the parameters $\Theta_1 = \dfrac{k_1 q_{1ec}^2}{b_1}$ and $\Theta_2 = \dfrac{k_2 q_{2ec}^2}{b_2}$ has to be noted. The magnitude of these parameters is of most importance for the possibility of transition of q from one area into the other. It also influences the time q stays in one of these areas. In other words, parameters q_{1ec} and q_{2ec} determine not only the possibility of a transition of a dynamic system from one characteristic state into another, but also determine the relative time this system stays in one of these states. In particular, for $\Theta_1 = \Theta_2 = \Theta > 0$, when Θ increases, the difference between parameters $f(q_{1m})$ and

$f(q_{2m})$ grows ($f(q_{1m}) < f(q_{2m})$), and the first area under the distribution curve is minimised (Figure 5.7 (b)). Practically, this implies that although central control is capable of minimising the time that the system is in the first area (state), it is unable to prevent the possibility of a system's transition to this characteristic state (although this probability is low, it nevertheless exists). In other words, the stabilisation activity of the control centres emerges mainly in the strategic plan, i.e. over the medium-term and long-term future, while in the short-term this initiative could turn out to be ineffective. Similarly, when central control reduces (Θ decreases), the difference between parameters $f(q_{1m})$ and $f(q_{2m})$ declines ($f(q_{1m}) < f(q_{2m})$), and the first area under the distribution curve enlarges (Figure 5.7 (c)). If there is no central control at all ($\Theta = 0$), distributions (5.50) and (5.51) coincide with distributions (5.46) and (5.47) (Figure 5.7 (a)). It is also worth noting, that for some values of the external parameters, distributions (5.50) and (5.51) turn out to be very close in shape to the single-mode distribution (5.39).

One more case is interesting: when the control centre takes the positions of those market participants, who are interested in declining q, which is equivalent to $-1 < \Theta < 0$ and to a change of the sign of q_{eC} in equation (5.48). Here, despite the tendencies of the impact of b_1 / k_1 and b_2 / k_2, changes in parameters Θ_1 and Θ_2 have the opposite impact. Indeed, for $-1 < \Theta_1 = \Theta_2 = \Theta < 0$, when Θ increases, the difference between the parameters $f(q_{1m})$ and $f(q_{2m})$ grows, but then $f(q_{1m}) > f(q_{2m})$ and the second area under the distribution curve is minimised (Figure 5.7 (d)). This implies, that under such central control it is possible to minimise the time when the system stays in the second area. In other words, ineffective activities of the control centre could result in reduced market stability in the short-term, medium-term and long-term. Despite the positions of the control centre, it is possible to conclude, that the impact on the dynamic system this centre has is determining. Parameters $f(q_{1m})$ and $f(q_{2m})$ can be used as characteristics of the position of the control centre.

The above analysis shows, that for the development of analytical models of the evolution of the spectral parameter q, multi-modal, and particularly two-modal, distributions can be applied. For the description of two-modal distributions, a composition of some two-digit δ-shape distribution (symmetrical or asymmetrical) can be used. This composition is characterised with the parameters q_{1e} and q_{2e}, and with some types of exponential distributions with corresponding powers. Together with the exponential components Θ_1, Θ_2 and m_1 ($m_1 = m + 1 = 2$ from equation (5.28)), there are other important parameters, determining the shape of such distributions, and representing the relative content of the discrete components in the composition: $|q_{1m} - q_c|$; k_1/b_1 and $|q_{2m} - q_c|$; k_2/b_2. If necessary, this composition can be organised in such a way, that for $m_1 < 2$ distributions will be leptokurtic, and for $m_1 \geq 2$ they will be mesokurtic. Also, it has to be pointed out, that the shape of the distributions (5.50) and (5.51) resembles distributions, characterising climatology laws for temperature distribution in different geographical areas. Usually, these distributions are also bi-modal and asymmetric, with supremums for winter and summer temperatures. This analogy allows us to consider that the real PDFs of values of the spectral parameter represent the sum of continuous distributions, characterising various states of the FX-market at different points of time.

5.6 CONCLUSION

The proposed theoretical model for distribution function $f(q,t)$, characterising changes of density probability of parameter q over time, allows us to obtain a full set of parameters, appropriate for finding practical solutions to the problem of identification of time series for currency exchange rates. Finding practical interpretations of the determining parameters of the system is one of the main aims of this experimental research, since only after this we will be able to turn our attention to justifying the use of the proposed theoretical model for the evolution of the spectral parameter.

The introduced analysis of the process of evolution of the spectral parameter, founded on the basis of asymptotic methods, is rather limited and can be used for simulation purposes only; the results have to be treated only as qualitative. This implication is evident at least because the stage of the evolution of the spectral parameter from one state of dynamic equilibrium into another is not deliberately considered in this research. Nevertheless, the proposed analysis is quite useful, as it has allowed us to determine the main characteristics of the process, which can be applied in further research. It would be natural to turn to the analysis of the behaviour of the set of continuous distributions, however, the behaviour of this set will be determined with some set of extra parameters, which are impossible to obtain in full at this stage of research with the proposed theoretical models (at least because this set of parameters, including normalisation coefficients, is still unknown to us). The problem of finding this set of parameters is among the priority tasks of this research field.

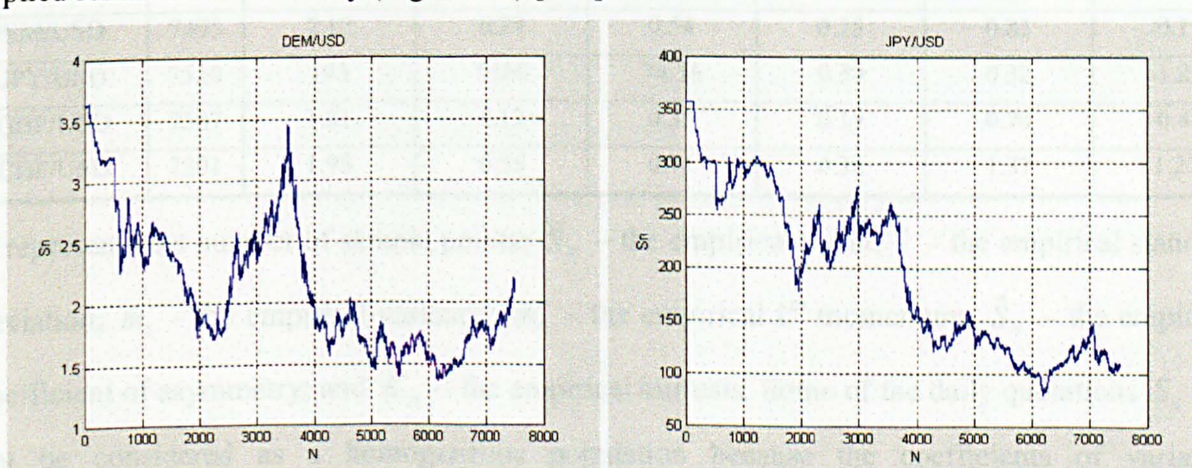
CHAPTER VI. Estimation of the Spectral Parameters for Time Series of Currency Exchange Rates

6.1 INTRODUCTION

This chapter presents the results of estimating of the spectral parameters of time series of currency exchange rates. Together with the preliminary handling and systematisation of the empirical data, this chapter presents the main stages of the method for experimental estimation of the spectral parameter (used for identification purposes), and the limits of applicability of the proposed method. Along with this, the issues of the estimation of the stochastic and chaotic components in the structure of the time series being considered are analysed, and a number of practical conclusions are made on the basis of a comparison of the results, including the justification of the use of diffusive-type stochastic equations, accounting for the existence of systematic changes (drifts), for analysing and simulation of the evolution processes of the spectral parameter q_q.

6.2 PRE-TREATMENT AND SYSTEMATISATION OF THE EXPERIMENTAL DATA

Time series of daily quotations of the exchange rates (DEM/USD, JPY/USD, GBP/USD, CHF/USD) for the period of January 1971 – June 2000 have been selected as the main data sets applied research for this study (Figure 6.1) [163].



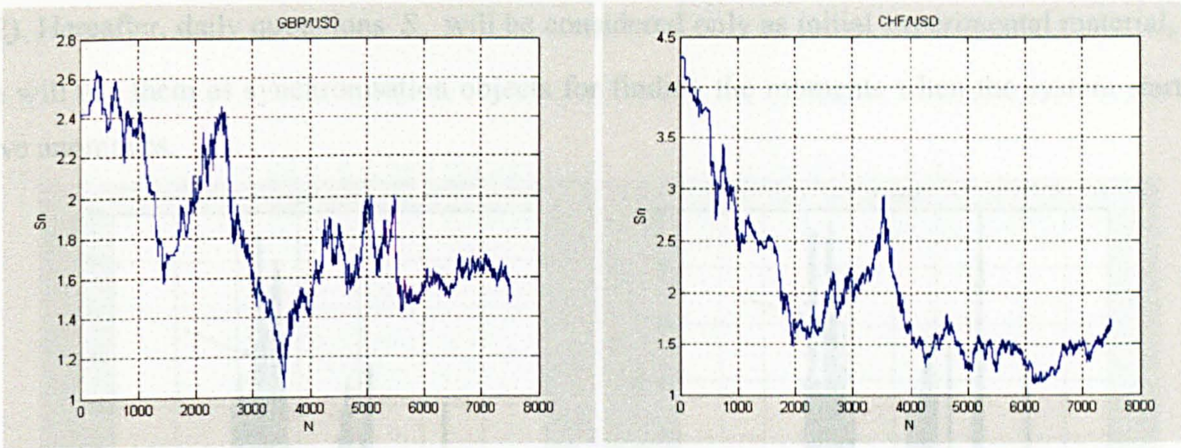


Figure 6.1. Daily Quotations of Currencies, Chosen as the Main Research Data Sets

From here onwards, the ordinate axis on all figures presented show the number of days (*N*), unless an alternative notion is introduced. Table 6.1 presents the cumulative data for the number of working days (*N*) for the considered years (1971-2000) for the specific example of the DEM/USD quotation. The values of the descriptive statistics on the univariate distribution of daily quotations S_n are presented in Table 6.2.

Table 6.1. Cumulative Data on the Number of Working Days for the Considered Years

Year	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980
N	245	496	746	996	1246	1498	1748	1998	2248	2499
Year	**1981**	**1982**	**1983**	**1984**	**1985**	**1986**	**1987**	**1988**	**1989**	**1990**
N	2750	3001	3251	3501	3751	4002	4262	4522	4782	5042
Year	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
N	5302	5564	5824	6084	6344	6605	6865	7125	7385	7495

Table 6.2. Descriptive Statistics for Sets S_n

Data	*N*	$\overline{S}_N$	$\hat{m}_2$	$\hat{s}$	$\hat{s}/\overline{S}_N$	$\hat{S}_N$	$\hat{K}_N$
DEM/USD	7495	2.12	0.29	0.54	0.25	0.83	-0.15
JPY/USD	7520	193	5560	74.56	0.39	0.32	-1.23
GBP/USD	7507	1.81	0.12	0.35	0.19	0.70	-0.48
CHF/USD	7501	1.98	0.55	0.74	0.38	1.37	1.22

N represents the number of sample points; $\overline{S}_N$ – the empirical mean; $\hat{s}$ – the empirical standard deviation; $\hat{m}_2$ – the empirical variance; $\hat{m}_k$ – the empirical k^{th} momentum; $\hat{S}_N$ – the empirical coefficient of asymmetry; and $\hat{K}_N$ – the empirical kurtosis. Some of the daily quotations S_n can not be considered as a homogeneous population because the coefficients of variation $\hat{s}/\overline{S}_N > 0.33$, so there is no need to test them for normality. The hypothesis of a normal distribution of S_n has to be rejected when there are high values of the coefficient of asymmetry (Figure

6.2). Hereafter, daily quotations S_n will be considered only as initial experimental material, and we will use them as synchronisation objects for finding the moments when the system starts to have anomalies.

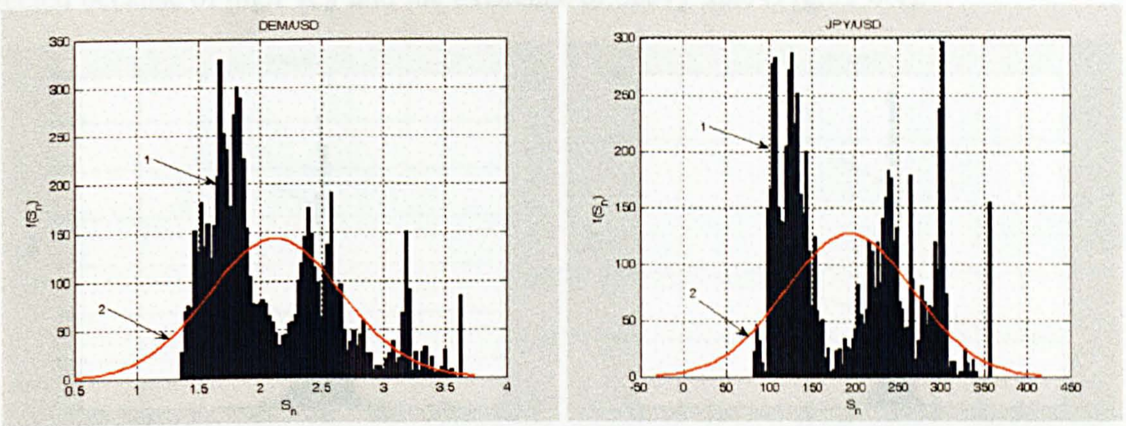


Figure 6.2. Density of Distribution of S_n for DEM/USD and JPY/USD

1: Density of Distribution of S_n; **2:** Approximation with Normal Distribution

In practice, profitability variables $s_n = S_n - S_{n-1}$, where $S_n = s_1 + ... + s_n$, and changes in profitability $s = (s_t)$, are usually of most interest for researchers (Figure 6.3). The profitability variables s_n are "more" homogeneous in comparison with S_n (Table 6.3), and are "more" characteristic parameters.

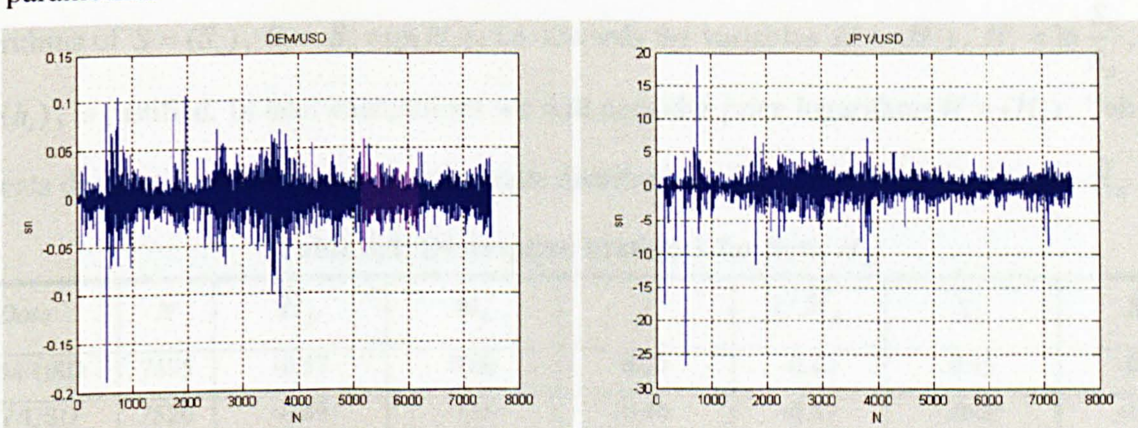


Figure 6.3. Profits s_n for DEM/USD and JPY/USD

Table 6.3. Descriptive Statistics for Sets s_n

Data	*N*	$\overline{s}_N$	$\hat{m}_2$	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$
DEM/USD	7494	$-2.11\cdot10^{-4}$	$1.94\cdot10^{-4}$	0.014	-0.48	10.20
JPY/USD	7519	-0.033	1.46	1.22	-2.00	49.33
GBP/USD	7506	$-1.18\cdot10^{-4}$	$1.14\cdot10^{-4}$	0.011	-0.27	5.28
CHF/USD	7500	$-3.54\cdot10^{-4}$	$2.11\cdot10^{-4}$	0.015	-0.09	12.38

It can be seen that we have negative mean values for profits of the currencies. However, at the absolute value, these means $\overline{s}_N << \hat{s}$, and thus s_n can be considered as equal to zero. Negative values for $\hat{S}_N$ indicate that empirical, and possibly real, density of distribution of s_n is asymmetric, and steeply dip from the right. The hypothesis of the normality of s_n distribution can still be rejected because of high $\hat{K}_N$ and the existence of heavy tails (Figure 6.4).

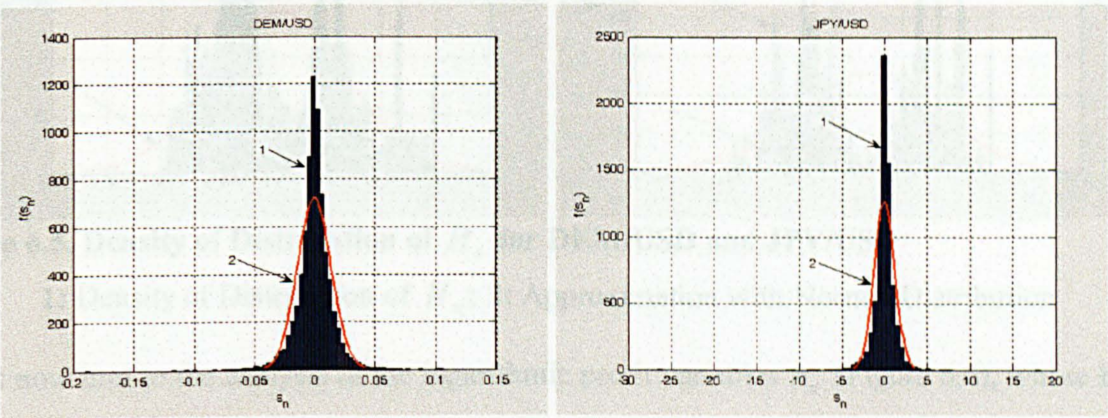


Figure 6.4. Density of Distribution of s_n for DEM/USD and JPY/USD

1: Density of Distribution of s_n; **2:** Approximation with Normal Distribution

Values representing relative changes $h_n = \ln S_n - \ln S_{n-1} = \ln \frac{S_n}{S_{n-1}}$, $H_n = h_1 + h_2 + ... + h_n$, $n \geq 1$ (which can be interpreted as logarithmic gains (returns) for $n \geq 1$) are of more economical significance than their real variables s_n. The interest towards logarithms and differences between logarithms of $S = (S_t)$, $S_t = S_0 \exp(H_t)$, i.e. towards the variables $H = (H_t)$, $H_t = \ln \frac{S_t}{S_0}$, and $h = (h_t)$, is justified. In later calculations we will consider price logarithms $H = (H_t)$. Table 6.4 presents descriptive statistics on the univariate distribution of H_n, with empirical mean $\overline{H}_N$.

Table 6.4. Descriptive Statistics for Sets H_n

Data	*N*	$\overline{H}_N$	$\hat{m}_2$	$\hat{s}$	$\hat{s}/\overline{H}_N$	$\hat{S}_N$	$\hat{K}_N$
DEM/USD	7495	-0.57	0.06	0.24	-0.42	0.45	-0.80
JPY/USD	7520	-0.69	0.16	0.40	-0.57	-0.02	-1.44
GBP/USD	7507	-0.30	0.03	0.18	-0.62	0.37	-0.50
CHF/USD	7501	-0.84	0.11	0.33	-0.39	0.78	-0.27

The use of parameters $H = (H_t)$ has to be admitted more appropriate in comparison with parameters $S = (S_t)$, since variables $\overline{H}_N$ are "more" homogeneous than $\overline{S}_N$, and the distributions H_n are more symmetric (in comparison with S_n). However, the variables H_n can not be rated as a homogeneous population because $|\hat{s}/\overline{H}_N| > 0.33$ (Figure 6.5).

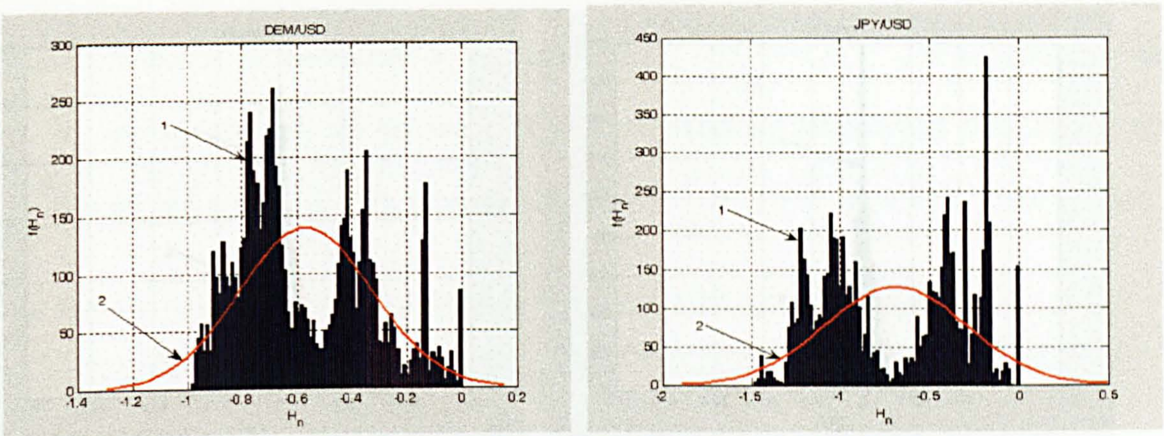


Figure 6.5. Density of Distribution of H_n for DEM/USD and JPY/USD

1: Density of Distribution of H_n; **2:** Approximation with Normal Distribution

Let us now turn to the analysis of the logarithmic profit variables h_n (Figure 6.6), whose behaviour is similar to the behaviour of the profit distribution s_n (Figure 6.3). However, statistical analysis of the variables h_n (Table 6.5 and Figure 6.7) shows that together with a mean value $\overline{h}_N$ which is nearly zero (still $\overline{h}_N << \hat{s}$), distributions of the variables h_n are more symmetric and less oblong, than the distributions of the profit variables s_n. The hypothesis of a normal distribution of the variables h_n can also be rejected because of high $\hat{K}_N$ (Table 6.5) and the existence of heavy tails in the distribution of h_n (Figure 6.7).

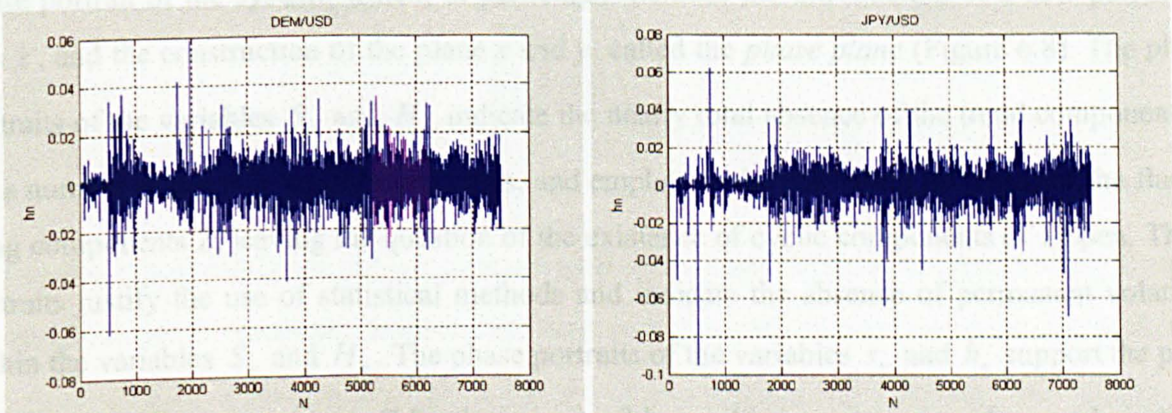


Figure 6.6. Profits h_n for DEM/USD and JPY/USD

Table 6.5. Descriptive Statistics for Sets h_n

Data	N	$\overline{h}_N$	$\hat{m}_2$	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$
DEM/USD	7494	$-6.57 \cdot 10^{-5}$	$4.26 \cdot 10^{-5}$	0.0065	-0.08	4.93
JPY/USD	7519	$-1.59 \cdot 10^{-4}$	$4.48 \cdot 10^{-5}$	0.0067	-0.94	13.49
GBP/USD	7506	$-5.15 \cdot 10^{-5}$	$3.76 \cdot 10^{-5}$	0.0061	-0.099	4.98
CHF/USD	7500	$-1.27 \cdot 10^{-4}$	$5.52 \cdot 10^{-5}$	0.0074	$-7.77 \cdot 10^{-4}$	4.02

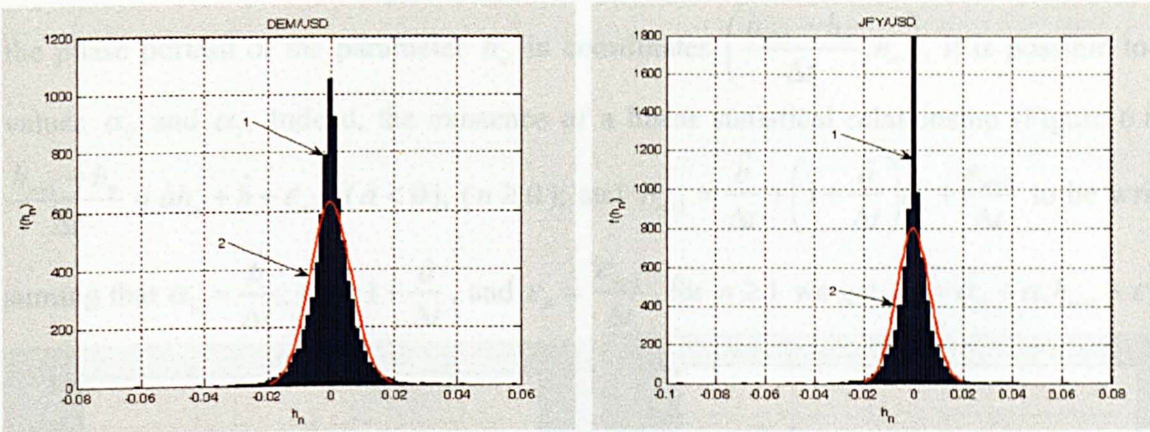


Figure 6.7. Density of Distribution of h_n for DEM/USD and JPY/USD

1: Density of Distribution of h_n; **2:** Approximation with Normal Distribution

In the time series h_n of the currency exchange rates, a fractional structure is observed for small values of the discreteness Δt, and tends to persists with increasing Δt, since it exists even in daily quotations of currencies. Thus, the use of fractional mathematical models, including the use of non-stationary fractional differential equation of time dependent order $q(t)$ (3.7) for the analysis of time series of currency exchange rates is well justified.

Turning to consider of the dynamic characteristics of the time series, it is worth paying attention to the key methods of visual analysis. Firstly, these methods are based on the construction of the phase portrait of the system, which, in particular, includes the introduction of a new parameter $y = \dot{x}$, and the construction of the plane x and y, called the *phase plane* (Figure 6.8). The phase portraits of the variables S_n and H_n indicate the nearly total absence of the trend components X for a number of possible equilibrium states, and emphasise the prevailing influence of the fluctuating components Z, leaving the question of the existence of cyclic components of Y open. These portraits justify the use of statistical methods and indicate the absence of permanent volatility within the variables S_n and H_n. The phase portraits of the variables s_n and h_n support the presence of fluctuating components Z for the one possible equilibrium state, unambiguously indicating the presence of statistical correlation between the considered parameter and the average speed of its change (the trend components X) in the processes, which is linear to a first approximation. Volatilities s_n and h_n can be considered as nearly constant within a given accuracy.

This result allows us to consider an (AR(1)) model: $h_n = \alpha_0 + \alpha_1 h_{n-1} + \varepsilon_n$ ($n \geq 1$), whose behaviour is completely determined by the noise parameters ε_n and initial conditions h_0. Using the phase portrait of the parameter h_n in coordinates $\left(\frac{h_{n+1}-h_n}{\Delta t}, h_n\right)$, it is possible to estimate values α_0 and α_1. Indeed, the existence of a linear statistical relationship (Figure 6.8) allows $\frac{h_{n+1}-h_n}{\Delta t} = \hat{a}h_n + \hat{b} + \varepsilon_{n+1}$ ($\hat{a} < 0$), ($n \geq 0$); and $h_{n+1} = \frac{\hat{b}}{\Delta t} + \left(1 + \frac{\hat{a}}{\Delta t}\right)h_n + \frac{\varepsilon_{n+1}}{\Delta t}$ to be written. Assuming that $\alpha_0 = \frac{\hat{b}}{\Delta t}$, $\alpha_1 = 1 + \frac{\hat{a}}{\Delta t}$, and $\varepsilon_n = \frac{\varepsilon_{n+1}}{\Delta t}$, for $n \geq 1$ we get $h_n = \alpha_0 + \alpha_1 h_{n-1} + \varepsilon_n$.

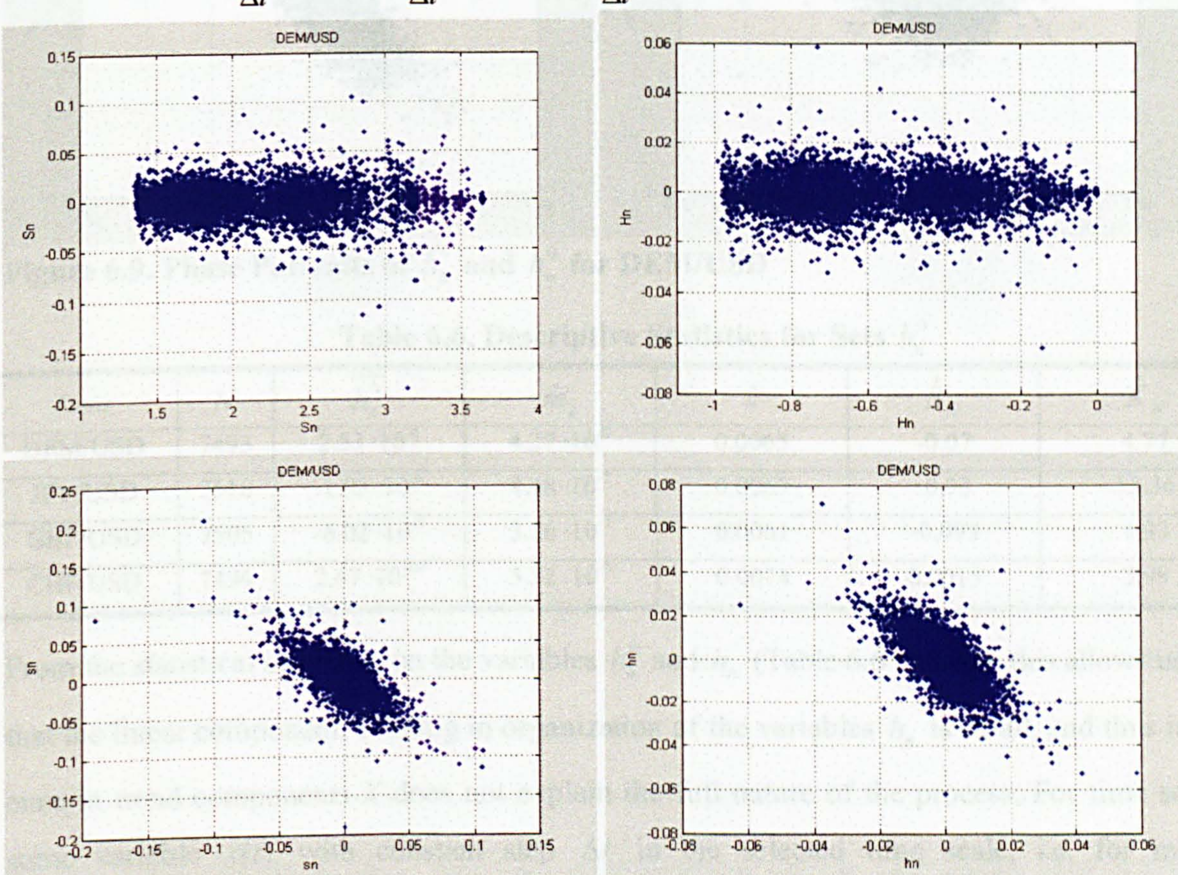


Figure 6.8. Phase Portraits of S_n, H_n, s_n, h_n for DEM/USD (for complete time series)

As a result, it is possible not only to estimate parameters k and b (using coordinates $q, \dot{q}$), which are included into the proposed model of the evolution of q, but also to organise the new statistic h_n^0, characterising the random component ε_n of the process. If, following the results that $Q_n \equiv R_n / S_n$ of the *R/S Analysis*, we can introduce the statistic $v_n = Q_n / \sqrt{n}$ for the AR(1) model (by variables of h_n), and it is possible to see that the values v_n rise with increasing n. But, this does not necessarily imply that we are dealing with models with fractional Gaussian noise, where $H > 0.5$, because this increase could be caused not by the fractionality of $(\varepsilon_n)_{n\geq 0}$, but by the existence of a linear relationship in the initial model $h_n = \alpha_0 + \alpha_1 h_{n-1} + \varepsilon_n$ ($n \geq 1$). For determining the stochastic nature of sequence ε_n, instead of using variables $h = (h_n)_{n\geq 1}$, we have to use the variables $h_n^0 = h_n - (\alpha_0 + \alpha_1 h_{n-1})$, $(n \geq 1)$, (Table 6.6 and Figure 6.9) and on their basis to organise a new parameter v_n^0.

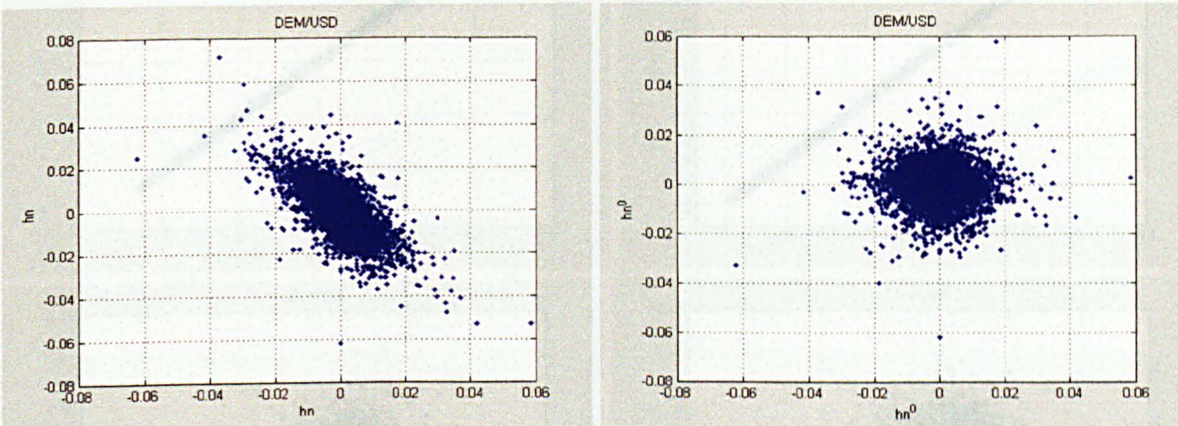


Figure 6.9. Phase Portraits of h_n and h_n^0 for DEM/USD

Table 6.6. Descriptive Statistics for Sets h_n^0

Data	N	$\overline{h_n^0}$	$\hat{m}_2$	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$
DEM/USD	7493	$2.53 \cdot 10^{-4}$	$4.27 \cdot 10^{-5}$	0.0065	-0.07	4.77
JPY/USD	7518	$-1.93 \cdot 10^{-8}$	$4.48 \cdot 10^{-5}$	0.0067	-0.93	13.36
GBP/USD	7505	$-6.02 \cdot 10^{-5}$	$3.76 \cdot 10^{-5}$	0.0061	-0.093	4.93
CHF/USD	7499	$2.67 \cdot 10^{-10}$	$5.52 \cdot 10^{-5}$	0.0074	0.0015	3.99

From the statistical inference on the variables h_n^0 and h_n (Table 6.6 and 6.5) this allows us to say that the linear component, existing in organization of the variables h_n is small, and thus its presence in trend components X does not explain the full nature of the process. For time series of some variable $x(t)$ with constant step Δt in the selected time scale, i.e. for moments $t_i = t_0 + (i-1)\Delta t$, $i = 1,\ldots,N$, values x_i and x_{i+1} can be selected as variables. In some cases it turns out to be more convenient to work with these variables because the scales on both axes are the same. Figure 6.10 presents the phase portraits of the variables S_n, H_n, s_n, h_n in this considered axes, constructed for DEM/USD for the completed time series. The time series S_n and H_n can be generally rated as deterministic sequences with small noise. The application of non-linear dynamic methods is not very effective just as they are, because all points are compressed along the line and we will have to distinguish very small scales. This requires very large samples (significantly exceeding what is the available for this research). If the required large samples are available, there are other reasons that make this problem unsolvable, e.g. the unknown complexity of the dynamics. This requires further consideration for these methods of statistical analysis.

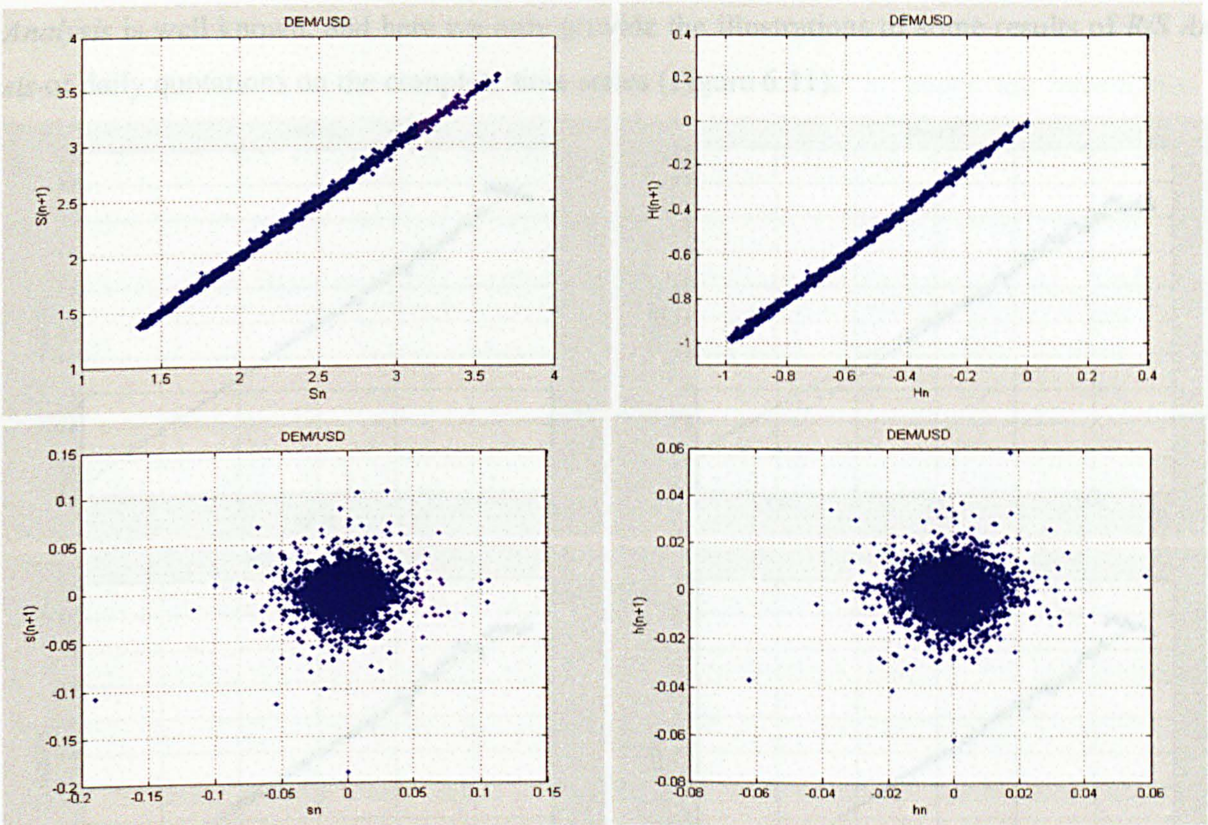


Figure 6.10. Phase Portraits of S_n, H_n, s_n, h_n for DEM/USD in Coordinates ($x_i; x_{i+1}$)

Along with the conclusion that the best forecast for tomorrow is the price for today, we again consider the AR(1) model in the form $H_n = \alpha_0 + \alpha_1 H_{n-1} + \varepsilon_n$ $(n \geq 1)$ for the analysis of the evolution of the variable H_n. Using the phase portrait of the variable H_n in coordinates $(H_n; H_{n+1})$, it is possible to estimate values α_0 and α_1, the random component of the process, and to organise the new statistics H_n^0 and h_n^0, characterising more thoroughly this component. Since: $h_{n+1} = (\alpha_0 + \alpha_1 H_n + \varepsilon_{n+1}) - (\alpha_0 + \alpha_1 H_{n-1} + \varepsilon_n) = \alpha_1 (H_n - H_{n-1}) + (\varepsilon_{n+1} - \varepsilon_n) = \alpha_1 h_n + \varepsilon_{n+1}$ we get $h_n = \alpha_1 h_{n-1} + \varepsilon_n$, $(n \geq 1)$. However, in this case all these differences turned out to be negligible and nearly equal to the estimates h_n^0 (Table 6.6). This allows us to reject of the assumption of the dominating influence of a linear relationship in the organisation of sequences of parameters h_n (Figure 6.10).

We will use approaches/techniques in order to eliminate the linear relationships for the other statistical methods. So, in *R/S Analysis* the value $\frac{1}{n} H_n$, where $H_n = \sum_{k=1}^{n} h_k$, will be considered as a good estimate of the mean value of the variables $h_1, h_2, ..., h_n$. The $H_k - \frac{k}{n} H_n$ will be used as a standard deviation within the last *n* measurements, and can also be considered as some numerical sequence, equivalent to the initial one, after linear de-trending. The methodology of *R/S*

Analysis is well known, and here we only provide the illustrations of some results of *R/S Analysis* of daily quotations on the complete time series (Figure 6.11).

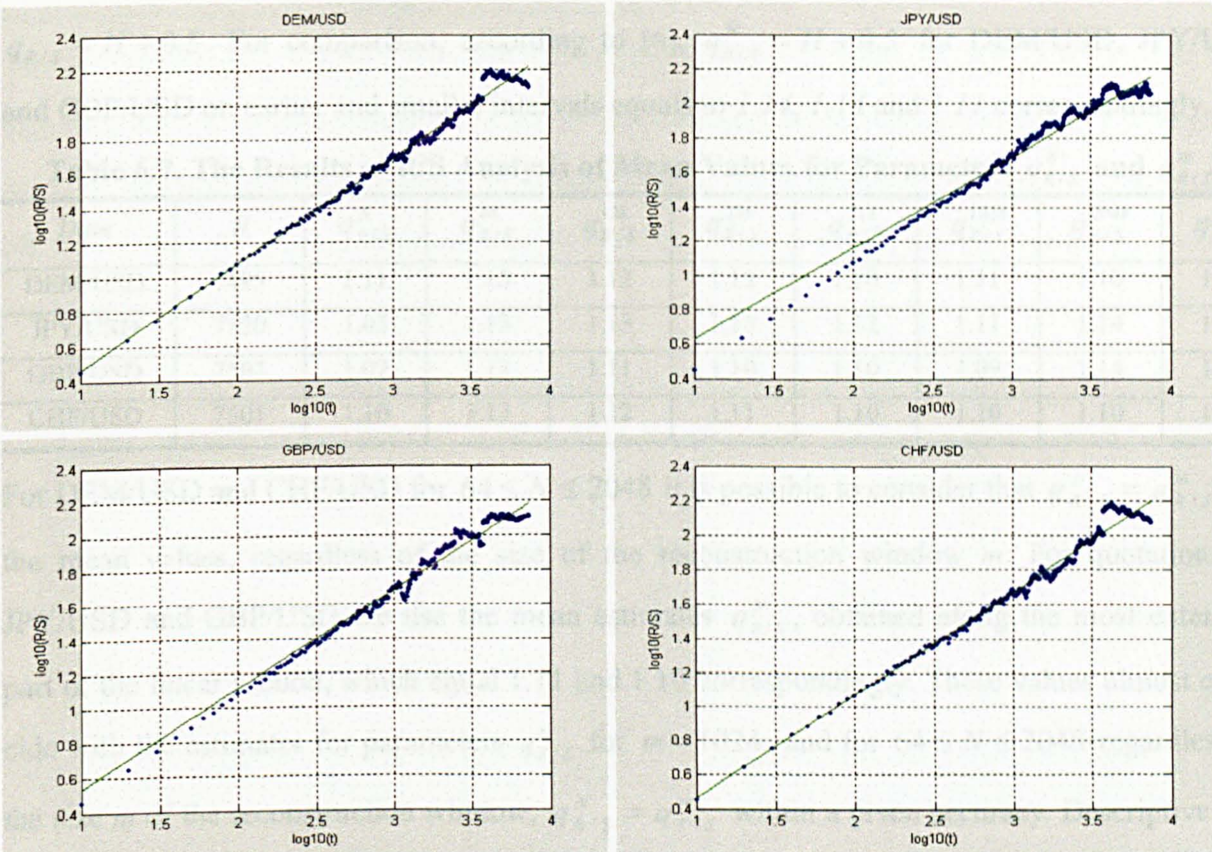


Figure 6.11. R/S Analysis of Analysed Daily Quotations (for complete time series)

For large N, the methodology of *R/S Analysis* for currency quotations is faced with the problem of choosing a linear section by which to estimate H. This is clear for the JPY/USD and GBP/USD quotations, where for large N, values of R_n / S_n stabilise, and the value of parameter H turns out to be evidently conservative for the complete sequence (Figure 6.11). For most quotations for intermediate values N (in particular $N \sim 1000$), local stabilisation of the values of R_n / S_n can also be observed, which implies the existence of a cyclic component Y in the considered data. The cyclic period lasts for about four years, and is usually attributed to the four-years period of presidential government in the US. It is possible to conclude, that the problem of choosing this linear section is mostly difficult for $N > 1000$. Using the ideas of reconstruction of the attractor, we choose some dimension m of the reconstruction window, and perform *R/S Analysis* within the corresponding window at the initial moment t_0. Afterwards, we repeat the *R/S Analysis* for the consecutive moment $t_1 = t_0 + n\Delta t$, where $n\Delta t$ is a discreteness of the new sequence, etc. Thus, for given values of m and Δt, we organise the new sequence $H^m_{R/S}(t,(m,\Delta t))$. Table 6.7 presents mean values for parameters $q^N_{R/S}$ of the daily quotations over the complete time series, and the results of *R/S Analysis* for finding the mean values for parameters $q^m_{R/S}$, performed within the reconstruction window m under the assumption that $q_{R/S} = H + 0.5$. For comparison, according to [6], $q^N_{R/S} = H + 0.5$ for DEM/USD, JPY/USD and GBP/USD on earlier and smaller intervals equals to *1.14*, *1.14* and *1.11* correspondingly.

Table 6.7. The Results of R/S Analysis of Mean Values for Parameters $q^N_{R/S}$ and $q^m_{R/S}$

Data	N	$q^N_{R/S}$	$q^{64}_{R/S}$	$q^{128}_{R/S}$	$q^{256}_{R/S}$	$q^{512}_{R/S}$	$q^{1024}_{R/S}$	$q^{2048}_{R/S}$	$q^{4096}_{R/S}$
DEM/USD	7495	1.11	1.13	1.12	1.12	1.10	1.11	1.10	1.17
JPY/USD	7520	1.03	1.13	1.13	1.13	1.12	1.11	1.14	1.11
GBP/USD	7507	1.07	1.12	1.11	1.10	1.10	1.09	1.14	1.17
CHF/USD	7501	1.10	1.13	1.12	1.11	1.10	1.10	1.10	1.15

For DEM/USD and CHF/USD for $64 \le N \le 2048$ it is possible to consider that $q^N_{R/S} = q^m_{R/S}$ for the mean values, regardless of the size of the reconstruction window m. For quotations of JPY/USD and GBP/USD we use the mean estimates $q^N_{R/S}$, obtained along the most extended part of the linear section, which equal 1.11 and 1.10 correspondingly. These values almost coincide with the estimates for parameters $q^m_{R/S}$ for $m = 1024$, and for $64 \le N \le 2048$ regardless of the size m of the reconstruction window, $q^N_{R/S} = q^m_{R/S}$ within a given accuracy. Descriptive statistics of the $q^m_{R/S}$ distributions are presented in Table 6.8.

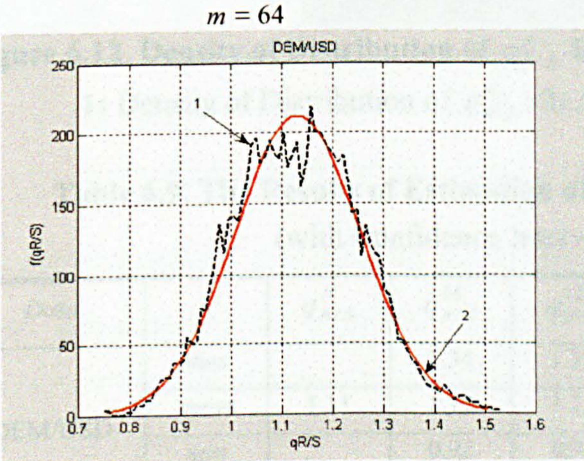
Table 6.8. Descriptive Statistics for Parameters $q^m_{R/S}$

Data	$q^m_{R/S}$	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$	$q^m_{R/S}$	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$
m	64				128			
DEM/USD	1.13	0.13	0.09	-0.27	1.12	0.10	0.11	0.03
JPY/USD	1.13	0.12	0.13	-0.17	1.13	0.10	0.03	0.03
GBP/USD	1.12	0.12	0.16	-0.02	1.11	0.10	0.13	0.48
CHF/USD	1.13	0.13	-0.003	-0.35	1.12	0.10	0.11	0.31
m	256				512			
DEM/USD	1.12	0.08	0.15	-0.22	1.10	0.07	0.01	-0.59
JPY/USD	1.13	0.07	-0.43	0.28	1.12	0.06	0.10	-0.63
GBP/USD	1.10	0.09	-0.32	-0.35	1.10	0.09	-0.26	-0.67
CHF/USD	1.11	0.10	0.02	-0.70	1.10	0.08	-0.54	-0.30
m	1024				2048			
DEM/USD	1.11	0.05	0.42	0.55	1.10	0.08	0.38	-0.90
JPY/USD	1.11	0.07	0.40	-0.66	1.14	0.05	-0.52	-0.71
GBP/USD	1.09	0.09	0.09	-0.54	1.14	0.09	-0.44	-1.25
CHF/USD	1.10	0.07	0.51	-0.26	1.10	0.09	0.55	-0.78
m	4096							
DEM/USD	1.17	0.03	-0.17	-0.86				

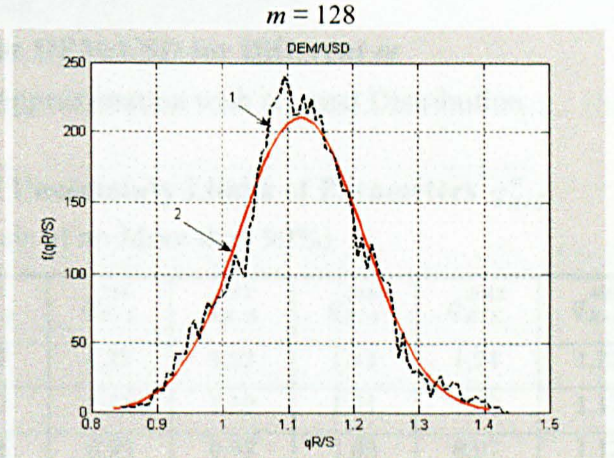
JPY/USD	1.11	0.02	0.33	-0.74
GBP/USD	1.17	0.04	-1.07	0.97
CHF/USD	1.16	0.02	-0.24	-0.82

The parameters $q^{m}_{R/S}$ can be attributed to a homogeneous population ($\hat{s}/q^{m}_{R/S} < 0.33$), and their distributions can be approximated (in the first approximation) to normal (Figure 6.12), what allows for an experimental estimation of the uncertainty limits in finding of these parameters (Table 6.9). Figure 6.13 presents the generalised results of mean values for $q^{N}_{R/S}$ and $q^{m}_{R/S}$ (according to daily quotations).

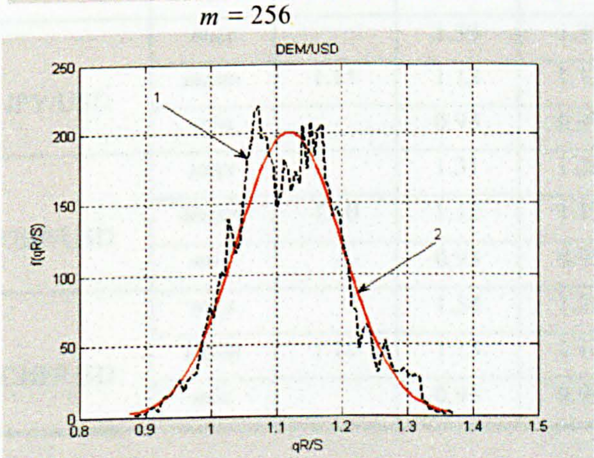
$m = 64$



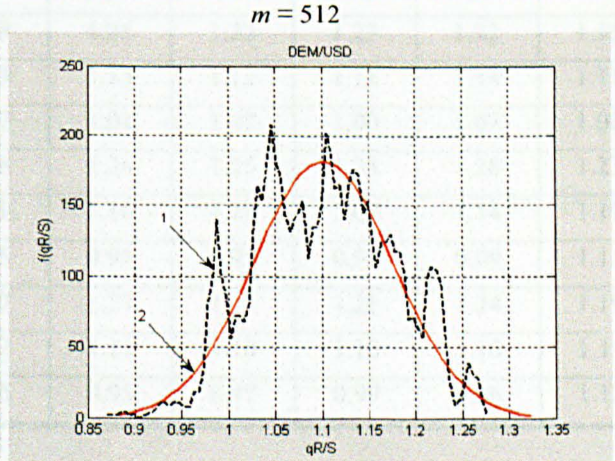
$m = 128$



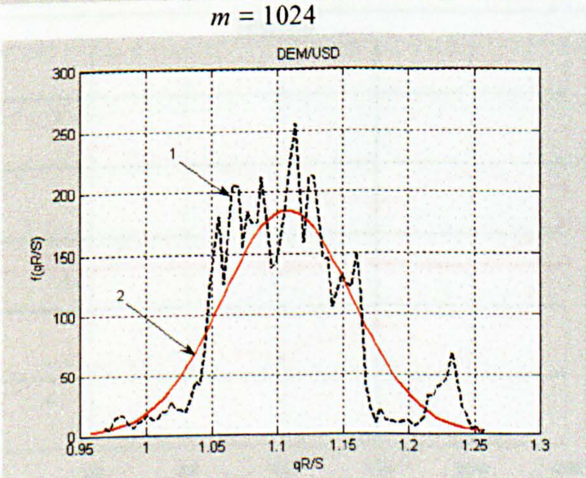
$m = 256$



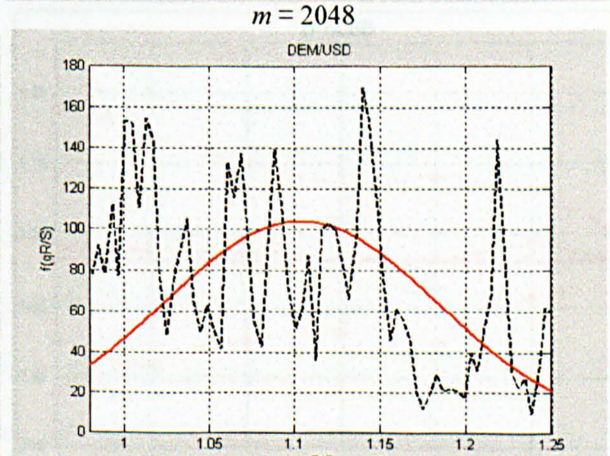
$m = 512$



$m = 1024$



$m = 2048$



$m = 4096$

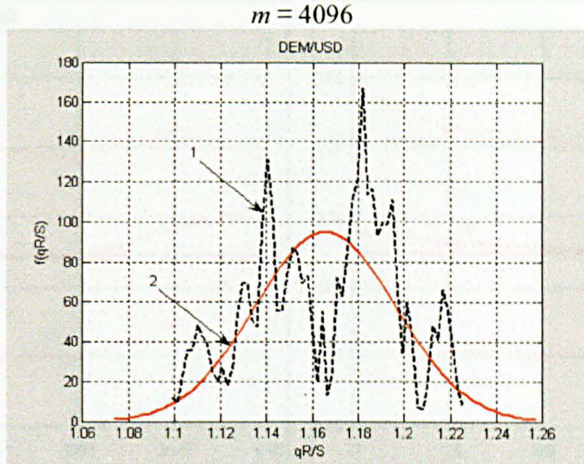
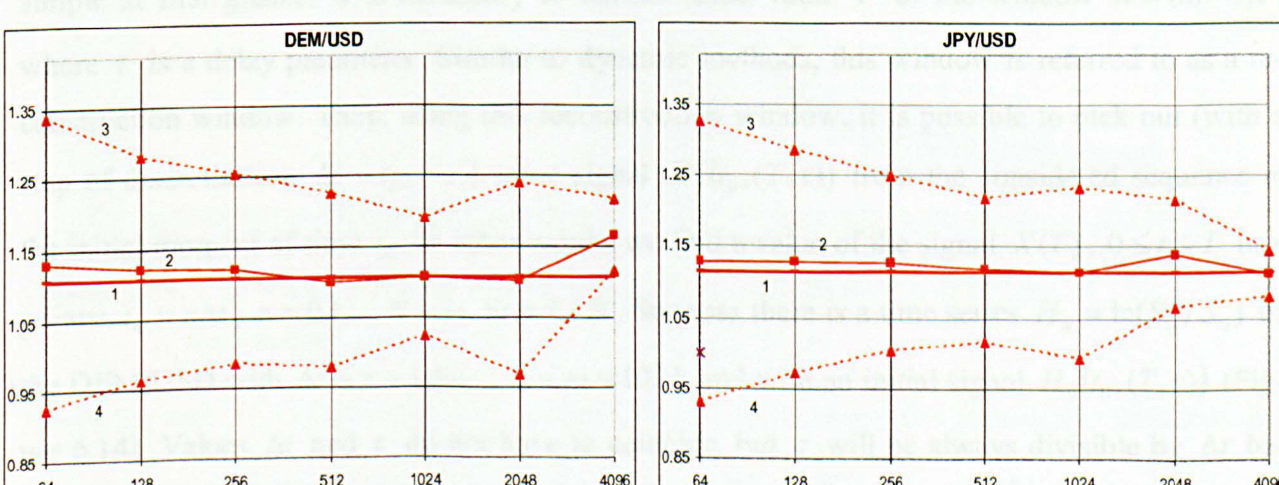


Figure 6.12. Density of Distribution of $q^m_{R/S}$ for DEM/USD for Different *m*

1: Density of Distribution of $q^m_{R/S}$; **2:** Approximation with Normal Distribution

Table 6.9. The Results of Estimation of Uncertainty Limits of Parameters $q^m_{R/S}$
(with Confidence Intervals of no More than 90%)

Data	*N*	$q^N_{R/S}$	$q^{64}_{R/S}$	$q^{128}_{R/S}$	$q^{256}_{R/S}$	$q^{512}_{R/S}$	$q^{1024}_{R/S}$	$q^{2048}_{R/S}$	$q^{4096}_{R/S}$
DEM/USD	*max*		1.34	1.28	1.25	1.22	1.19	1.24	1.22
	mean	1.11	1.13	1.12	1.12	1.10	1.11	1.10	1.17
	min		0.92	0.96	0.99	0.98	1.03	0.97	1.12
JPY/USD	*max*		1.33	1.29	1.25	1.22	1.23	1.22	1.15
	mean	1.11	1.13	1.13	1.13	1.12	1.11	1.14	1.11
	min		0.93	0.97	1.01	1.02	1.00	1.07	1.08
GBP/USD	*max*		1.31	1.28	1.26	1.25	1.23	1.28	1.23
	mean	1.10	1.12	1.11	1.10	1.10	1.09	1.14	1.17
	min		0.93	0.95	0.95	0.95	0.95	0.99	1.11
CHF/USD	*max*		1.34	1.27	1.27	1.23	1.21	1.24	1.19
	mean	1.10	1.13	1.12	1.11	1.10	1.10	1.10	1.15
	min		0.92	0.96	0.95	0.97	0.99	0.96	1.12



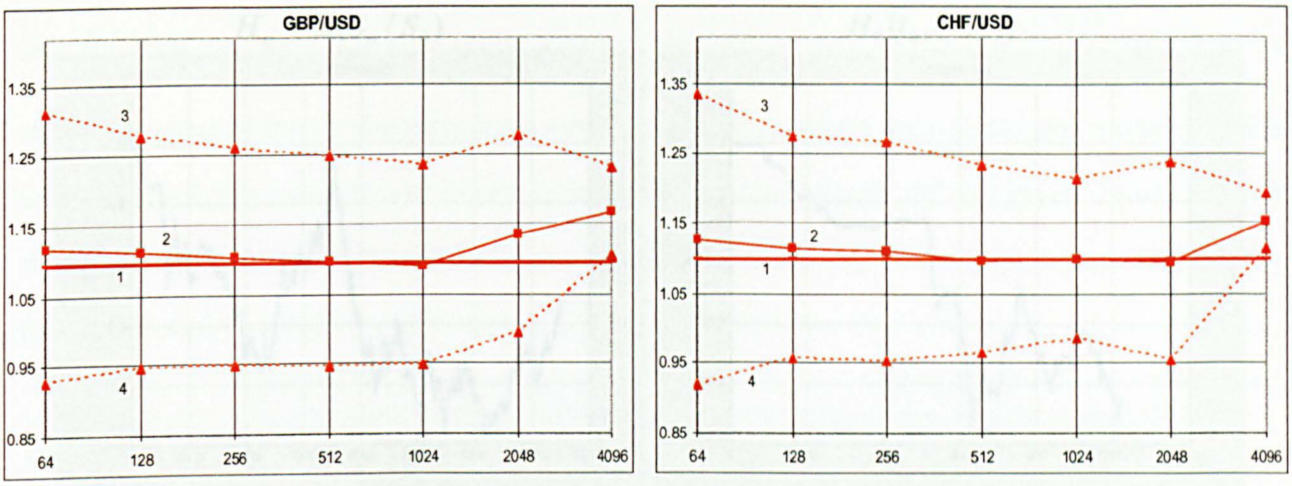


Figure 6.13. Generalised Results of Mean Values for $q_{R/S}^{N}$ and $q_{R/S}^{m}$

1: $q_{R/S}^{N}$; **2:** $q_{R/S}^{m}$; **3:** The Lower Estimate of Uncertainly of $q_{R/S}^{m}$; **4:** The Upper Estimate of Uncertainly of $q_{R/S}^{m}$ (dotted lines show the upper and lower boundaries)

For these currencies it is possible to consider (with high accuracy) that $q_{R/S}^{N} = q_{R/S}^{m}$ for the mean values, at least for $m \leq 1024$. Practical independence of the results of *R/S Analysis* from the size of the reconstruction window for $64 \leq m \leq 2048$ suggests that the results for daily quotations are reliable estimates of the dimension of the considered sets, and for further use of these results as quite reliable objects for comparison.

6.3 The Development of Methodology for Estimation of the Spectral Parameter, Used for Identification Purposes

The idea of finding the spectral parameter $q(t,(T,\tau))$ of an experimental time series looks rather simple at first glance. It is necessary to choose some value T of the window $w=(m-1)\tau$, where τ is a delay parameter. Similar to dynamic methods, this window is referred to as a reconstruction window. Then, using this reconstruction window, it is possible to pick out (with a step of discretisation $\Delta t = t_{i+1} - t_i$) some signal $X_0(t_0,(T,\tau))$ from the considered sequence at the initial moment of time t_0. In other words, we find a value of the signal $X(T)$, $0 \leq t \leq T$ in n points: $t_n = n\Delta t$, $n = 0,1,\ldots,N-1$; $\Delta t = T/N$. Suppose there is a time series $H_n = \ln(S_n / S_0)$ of the DEM/USD with $\Delta t = \tau = 1\,\text{day}$, $N = m = 1024$ and with an initial signal $H_0(t_0,(T,\tau))$ (Figure 6.14). Values Δt and τ do not have to coincide, but τ will be always divisible by Δt because of the discrete homogeneity of the initial sequence.

$H_n = \ln(S_n / S_0)$ $\qquad\qquad$ $H_0(t_0,(T,\tau))$

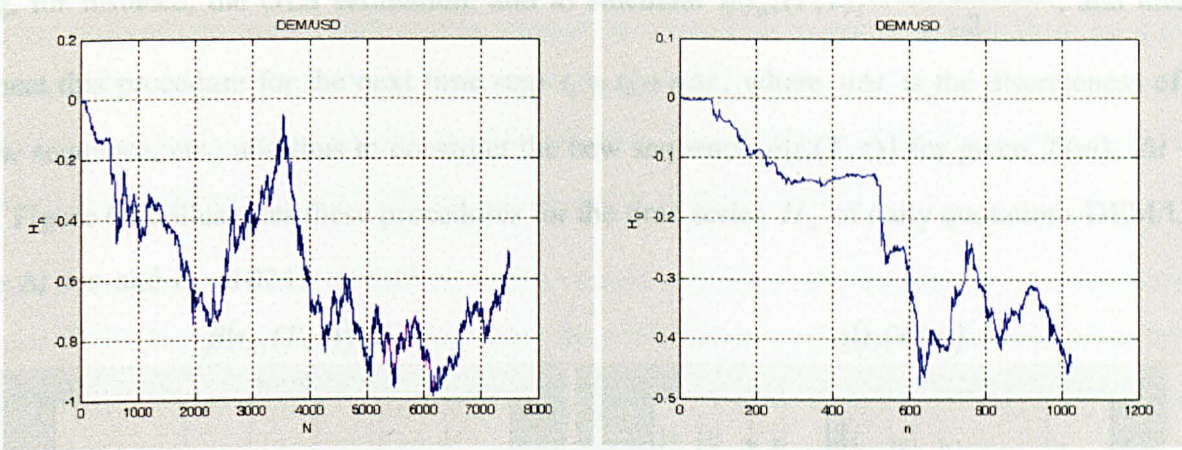


Figure 6.14. Time Series $H_n = \ln(S_n / S_0)$ and Initial Signal $H_0(t_0,(T,\tau))$ for DEM/USD with $\Delta t = \tau = 1\,\text{day}$, $m = 1024$

Having $X_n = X(t_n)$, we use *m* counts of the signal $X(T)$ to obtain the same number of values of the function $\hat{X}(f)$: $f_n = \dfrac{n}{m\Delta t}$, $n = 0,1,\ldots,m$, i.e. using a FFT of the initial signal $X_0(t_0,(T,\tau))$, it is possible to obtain the signal $\hat{X}_0(f,(T,\tau))$ in the frequency domain, and find the spectral power $M_{X_0}(f,(T,\tau))$ of this signal. Due to symmetry, for even *m* and for real values, we use only $m/2$ counts (the right hand side of spectrum): $n = 1 + m/2 \ldots m$ (Figure 6.15). For convenience and clearness, we put on the *X*-line here and after the corresponding values of $n \sim f_n$ or $\log n \sim \log(f_n)$, $n = 1,2,\ldots,m/2$, instead of $f_n = \dfrac{2n}{m\tau}$ or $\log f_n = \log\left(\dfrac{2n}{m\tau}\right)$.

$\hat{H}_0(f,(T,\tau))$ $\qquad\qquad$ $M_{H_0}(f,(T,\tau))$

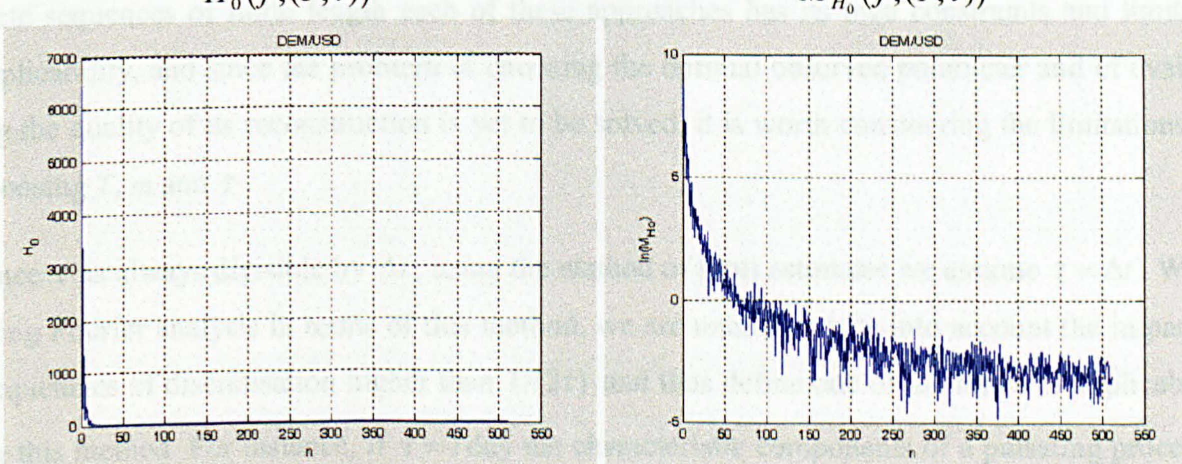


Figure 6.15. The Kind of Initial Signal $\hat{H}_0(f,(T,\tau))$ and its Spectral Power $M_{H_0}(f,(T,\tau))$ for Finding $\beta(t_0,(T,\tau))$

Assuming that $M_{X_0}(f,(T,\tau)) \sim f^{-\beta}$, where $\beta(t_0,(T,\tau))$ is a spectral parameter, characterising the power spectral dimension of the system, it is possible to find $\beta(t_0,(T,\tau)) = 2q(t_0,(T,\tau))$, using, for instance, the OLS estimation; and to calculate $q(t_0,(T,\tau)) = \dfrac{\beta(t_0,(T,\tau))}{2}$; and then to repeat this procedure for the next time step $t_1 = t_0 + n\Delta t$, where $n\Delta t$ is the discreteness of the new sequence, etc.; and thus to construct the new sequence $q(t,(T,\tau))$ for given $T(m)$, Δt and τ. Figure 6.16 illustrates these procedures for the time series H_n of daily quotations DEM/USD for $\Delta t = \tau$ and $m = 1024$.

$\beta(t_0,(T,\tau))$ $\qquad\qquad\qquad\qquad$ $q(t,(T,\tau))$

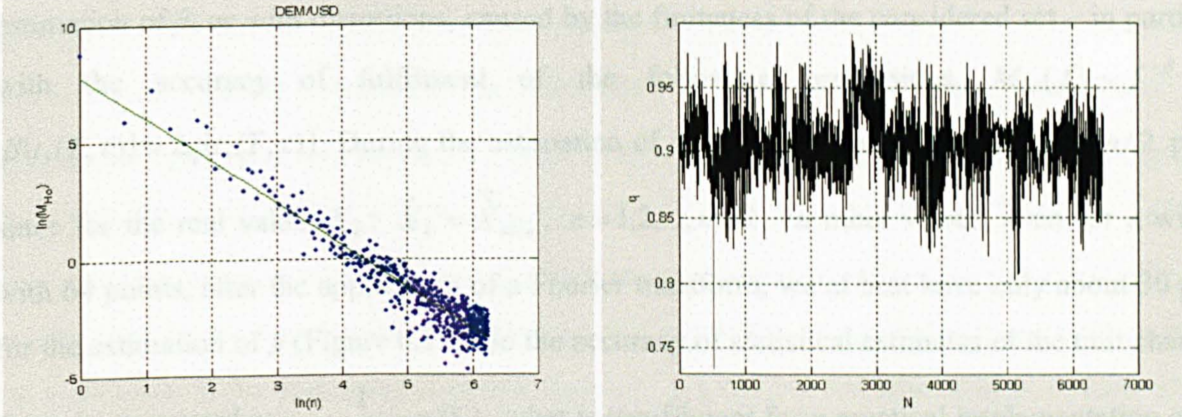


Figure 6.16. Estimation of $\beta(t_0,(T,\tau))$ and the New Constructed Sequence $q(t,(T,\tau))$

However, the simplicity of this method is deceptive since its initial idea unites two approaches for studying time series – the approach based on the methods of non-linear dynamics (in particular, those based on the idea of the reconstruction of the attractor), and the other approach, based on statistical methods of analysing the time series with Fourier analysis. When considering discrete sequences of finite length each of these approaches has its own constraints and limits of applicability, and since the problem of choosing the optimal observed parameter and of evaluating the quality of its reconstruction is yet to be solved, it is worth considering the limitations for choosing T, m and τ.

Since τ is always divisible by Δt, using the method of limit estimates we assume $\tau = \Delta t$. While using Fourier analysis in terms of this method, we are unable to take into account the impact of frequencies of discretisation higher than $1/(2\tau)$ and thus define one of the limits of applicability for this method. For instance, if $\tau = 1$ day the characteristic components of a pulsating processes with a period less than two days could not be obtained using this method. It is necessary to minimise τ, i.e. to use time series with a smaller discreteness Δt. However, it is not possible to minimise τ beyond the boundaries of this method. For correct estimation of the average speed of the diffusive process, much longer time intervals are needed. These time intervals have to be of a higher order than those, during which the change of the pulsating components of this process occurs. According to Table 2.1, when the time interval is smaller than a quarter of an hour, the use of these considered methods for most of the currencies is inappropriate. Thus, for a lower estimate, the considered method can be used for the analysis of currency exchange rates when the time interval 2τ is not smaller than 30 minutes. This does not allow this method to be considered as a tool for technical analysis, i.e. for active short-term trades.

When m reduces, the task of finding of the correct estimates of β becomes more complicated. These difficulties are associated with either the problems of instability in the algorithm for the estimation of β, or with distortions, caused by the finiteness of the considered set – in particular, with the accuracy of fulfilment of the following proportions $M_X(f) \sim f^{-\beta}$ and $\beta(t,(T,\tau)) = 2q(t,(T,\tau))$. During the estimation of β we obtain not m, but, at best, $m/2$ points, since for the real values X_k: $\hat{X}_k = \bar{\hat{X}}_{m-n}$, $n = 1,2,...,m/2$. In other words, even for a window with 64 points, after the application of a Fourier transform, we at best have only about 30 points for the estimation of β (Figure 6.17). So the accuracy of statistical estimates of the unit change in β can be measured as $\sim \frac{1}{\sqrt{m/2}} \approx 0.2$, what is insufficient for a practical implementation even if the method for the estimation of β is robust.

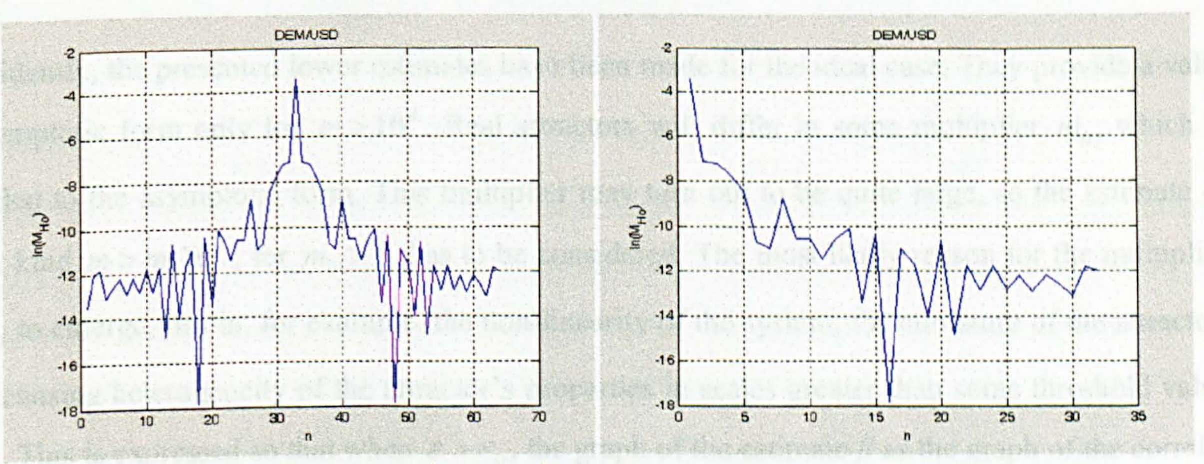


Figure 6.17. Spectral Power $M_{H_0}(f,(T,\tau))$ for $\Delta t = \tau$, $m = 64$

Finding the exact threshold of the stability failure of methods of β estimation on real data with the use of statistical methods is rather difficult. This task becomes more complicated when it is not known at what time (while m declines) the initial set distorts, that is looses nearly all its major attributes, and we need to turn to the analysis of some other set. This implies understanding the accuracy of ratios $M_X(f) \sim f^{-\beta}$ and $\beta(t,(T,\tau)) = 2q(t,(T,\tau))$ for small m. The necessary lower estimates can be easily obtained experimentally. Nevertheless, let us try doing their estimation with the methods of non-linear dynamics. Practically, there is a problem of the estimation of the dimension using a finite set of points, belonging to this set. To solve this problem, it is assumed that the set is scale invariant, and the correct estimation can be made without limiting $\varepsilon \to 0$. The change of scale, where invariance is noticeable, is not significantly large, and points are uniformly spread around the set; which has dimension *d*. So, how many points are needed to estimate this dimension?

Similar to random processes, it is possible to assume that to find (for instance) the scale invariance of the set, this scale needs changing to be at least ten times greater. If the considered set is a cube of size *d* and side *a*, the distance to the nearest point has to be $\sim 0.1a$. That is, each point takes the size $\sim (0.1a)^d$, and for filling in this cube, $m \cong \dfrac{a^d}{(0.1a)^d} = 10^d$ points are needed. The considered data sets do not represent a cube, although generally asymptotic form still has to persist. Then, for a quantitative estimation for $d = 1.5$, in an ideal case no less then $10^{1.5} = 32$ points are needed; and no less then 100 points are needed for an estimation for $d = 2$. While using the FFT for the reconstruction window *m* of size 16, 32 and 64, difficulties could emerge for the estimation of the dimension *d* greater than 1.2, 1.5 and 1.8 correspondingly. In case of a univariate process ($d = 2 - H$, $H \in (0;1]$), for windows of size $m \geq 128$, there are no difficulties of this kind in the ideal case.

Evidently, the presented lower estimates have been made for the ideal case. They provide a valid asymptotic form only for $m > 10^d$. Real attractors will differ in some multiplier m_0, which is added to the asymptotic form. This multiplier may turn out to be quite large, so the estimate of the kind $m > m_0 10^d$, for $m_0 > 1$, has to be considered. The most likely reason for the multiplier m_0 to emerge, lies in, for example, the non-linearity of the system, the curvature of the attractor, or causing heterogeneity of the attractor's properties in scales greater than some threshold value ε_0. This is expressed so that when $\varepsilon > \varepsilon_0$, the graph of the estimate β as the graph of the correlation integral, are loosing linear, and the vector in the reconstruction can not be considered as a tangent. The theoretical estimation of the parameter m_0 for a real time series is a separate problem, going beyond this research. Therefore, for estimations we use lower the ideal estimates of the value *m*, supplemented with an experimental check on the obtained results. With regards to time series for currency exchange rates and following the previous discussion on estimations, it is possible to state that the use of this method for $m < 64$ is barely justified. The parameters *m* and τ are critical for the estimation of the limits of applicability of the considered method (at least for small T), as opposed to the dynamic methods, where the final result is influenced by the size of the reconstruction window $w = (m-1)\tau$.

Another type of misrepresentation emerges when m and T are sufficiently large, and, instead of studying the structure of the set, we start analysing the structure of its folds. Since a theoretical determination of the upper estimates is complicated, here we only focus on an experimental estimation, measuring $q(T,\tau)$ for different m, and looking for those $q(T,\tau)$, that produce results almost independent of m. By doing this, we assume that there are no serious upper limitations for non-chaotic systems. If chaotic and cyclic components exist in the processes, we assume that on large scales of reconstruction the object is likely to look like the object of higher dimension d than the actual object (q are smaller than the real values). Finally, if trend components exist in the processes, it is assumed the object is likely to look like the object of a smaller dimension d (q are greater than the real values).

When considering methods for estimating the parameter $q(t,(T,\tau))$ with the use of the $M_X(f) \sim f^{-\beta}$ and $\beta(t,(T,\tau)) = 2q(t,(T,\tau))$ ratios, it is assumed that for each time series of currency exchange rates there are on average 10^4 experimental points available. The size of the reconstruction window has to average 10^3 experimental points, i.e. the choice of the basic parameter as $m_q = 1024$ points is justified, for this process at least. After taking the logarithm, and assuming that $f_n = \dfrac{2n}{m\tau}$, $n = 1 + m/2 \ldots m$, we get: $\log M_X\left(\dfrac{2n}{m\tau}\right) \sim -\beta \log\left(\dfrac{2n}{m\tau}\right) + \alpha_1$ or $\log M_X(n) \sim -\beta \log(n) + \alpha$. In logarithmic coordinates $x = \log(n)$; $y = \log M_X(n)$, experimental values group along the line with a slope coefficient $(-\beta)$, characterising the slope ratio (tangent) of this line to the X axis. Let us apply the OLS method for the experimental estimation of $(-\beta)$ for the window $m = m_q$, for DEM/USD quotations (Figure 6.18).

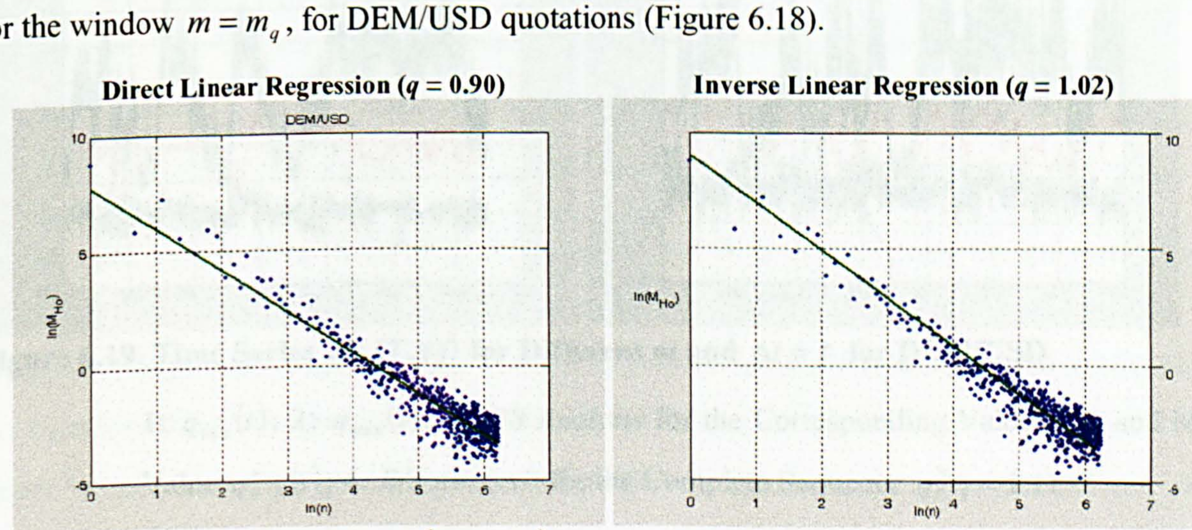


Figure 6.18. The Results of $\beta(t_0,(T,\tau))$ Estimation, for $\Delta t = \tau$ and $m = 1024$

The equation obtained from the direct linear regression $\hat{y} = a_{0(yx)}x + a_{1(yx)}$ is not algebraic, and thus does not allow x to be found directly from it, because this model was obtained through the minimisation of the sum of squared deviations along the Y axis. To find x (the inverse regression is $\hat{x} = \dfrac{y}{a_{0(xy)}} + a_{1(xy)}$), the sum of the squared deviations along the X axis then has to be minimised. As expected (Figure 6.18), the direct regression line is more flat towards the X axis, than the axis of $y = k_0 x + b$, and the inverse regression line is more steep. Since $a_{0(yx)} < k_0 < a_{0(xy)}$ and $a_{0(yx)} \neq a_{0(xy)}$, the composition of $a_{0(yx)}\left(\dfrac{1}{a_{0(xy)}}\right) \neq 1$, but equals to $\left(\dfrac{a_{0(yx)}}{a_{0(xy)}}\right) = \rho^2$, whose square root is the coefficient of a pairwise (cross) correlation between x and y. By the same reason, the slope coefficient of the ellipse of dispersion can be represented as $k_0 = \dfrac{a_{0(yx)}}{\rho}$ or as $k_0 = a_{0(xy)}\rho$. Assuming that the best estimate of q is the parameter $\dfrac{\beta}{2} = \dfrac{|k_0|}{2}$, it is possible to state, that regardless of the parameter m, the application of the equation of the direct regression line always produces a low estimate of the mean value $q^m_{\min}$, and the application of the equation of the inverse regression line – produces a high estimate of $q^m_{\max}$ (Figure 6.19). For comparison, Figure 6.19 presents the results of *R/S Analysis* of the same time series, obtained under the assumption $q_{R/S} = H + 0.5$ for both: the complete sequence $q^N_{R/S}$, and for the corresponding value of m – $q^m_{R/S}$. Similar results are observed for other quotations (Table 6.10, Figure 6.20).

$q(t,(T,\tau))$ for $m = 1024$,
$q^m_{\min} = 0.91$, $q^m_{\max} = 1.26$, $q^m_{R/S} = 1.11$

$q(t,(T,\tau))$ for $m = 512$,
$q^m_{\min} = 0.91$, $q^m_{\max} = 1.28$, $q^m_{R/S} = 1.10$

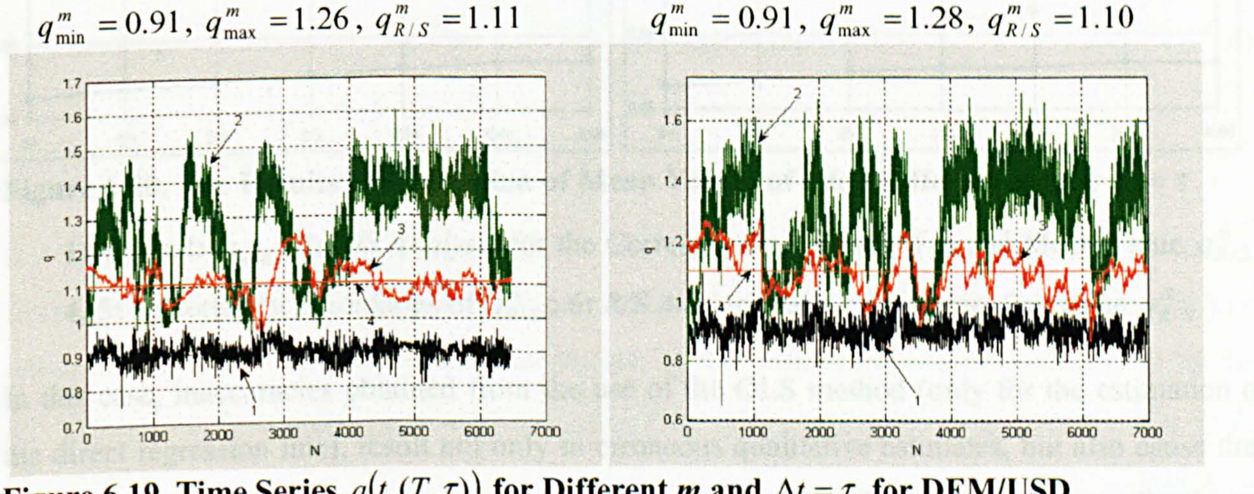


Figure 6.19. Time Series $q(t,(T,\tau))$ for Different m and $\Delta t = \tau$ for DEM/USD

1: $q_{\min}(t)$; **2:** $q_{\max}(t)$; **3:** *R/S Analysis* for the Corresponding Value of m and Mean Value $q^m_{R/S}(t)$; **4:** *R/S Analysis* for the Complete Sequence $q^N_{R/S} = 1.11$

Table 6.10. The Results of Estimation of Mean Values of q for Different Values of m

Data	$q^{N}_{R/S}$	$m=1024$			$m=512$		
		$q^{m}_{\min}$	$q^{m}_{\max}$	$q^{m}_{R/S}$	$q^{m}_{\min}$	$q^{m}_{\max}$	$q^{m}_{R/S}$
DEM/USD	1.11	0.91	1.26	1.11	0.91	1.28	1.10
JPY/USD	1.11	0.91	1.27	1.11	0.91	1.28	1.12
GBP/USD	1.10	0.91	1.29	1.09	0.90	1.30	1.10
CHF/USD	1.10	0.91	1.27	1.10	0.91	1.28	1.10

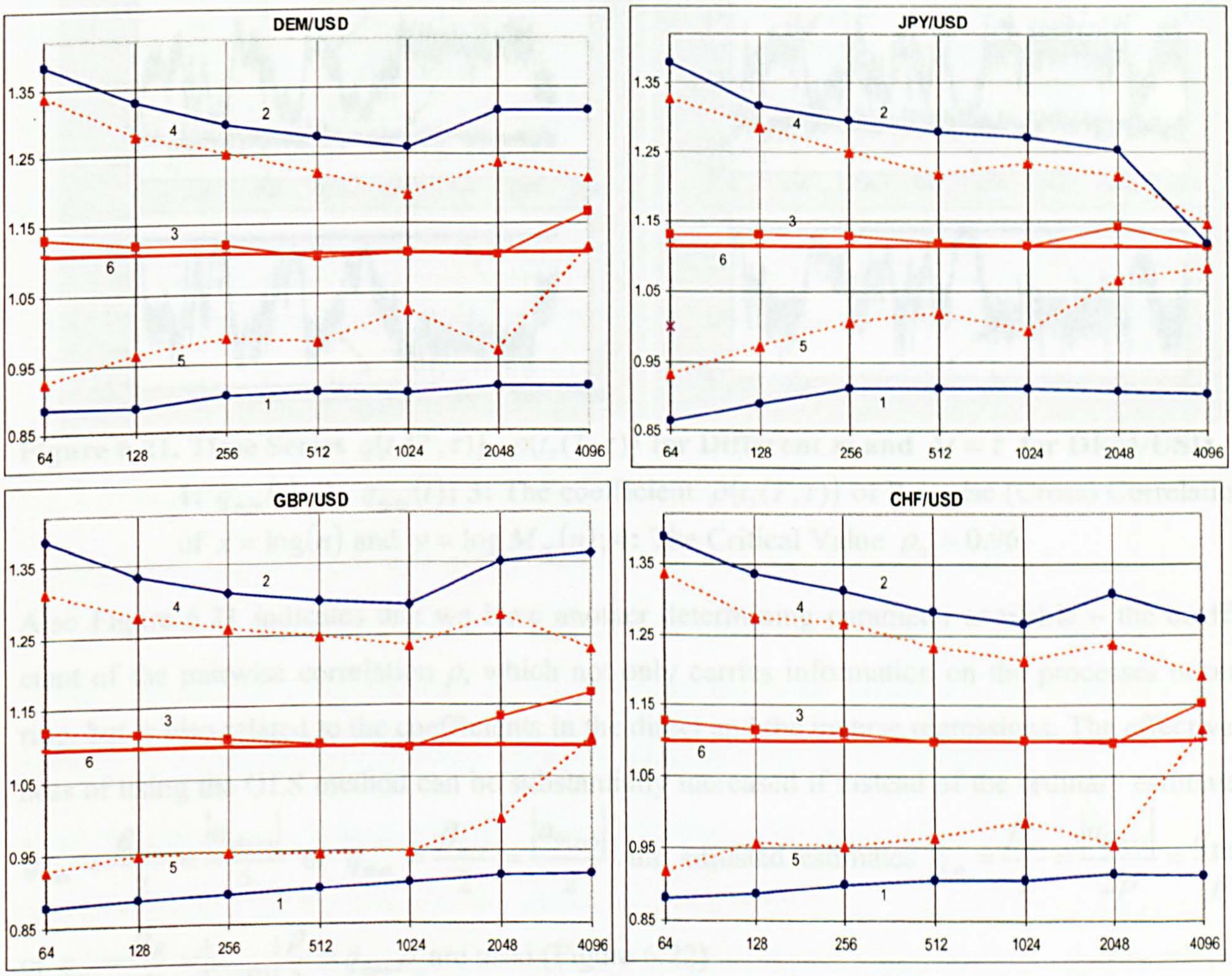


Figure 6.20. The Results of Estimation of Mean Values of q for Different m and $\Delta t = \tau$

1: $q_{\min}$; **2:** $q_{\max}$; **3:** *R/S Analysis* for the Corresponding Value of m and Mean Value $q^{m}_{R/S}$; **4, 5:** Uncertainty Boundaries of $q^{m}_{R/S}$; **6:** *R/S Analysis* for the Complete Sequence $q^{N}_{R/S}$

In this case, inaccuracies obtained from the use of the OLS method (only for the estimation of the direct regression line), result not only in erroneous qualitative estimates, but also cause dramatic misinterpretations in the nature of the processes occurring. For instance, the application of the equation of direct regression only results in the false conclusion that the single case of $q<1$ exists in quotations of currencies, which consequently allows for an erroneous resume of the dominance of the processes with "anti-persistence", etc.

Regardless of the size of the reconstruction window and the quotations of the currencies considered, in the majority of cases the coefficient $\rho(t,(T,\tau))$ of the pairwise (cross) correlation of $x=\log(n)$ and $y=\log M_X(n)$ is below the critical value $\rho_0=0.96$ (Subchapter 4.3), which suggests that the OLS method is ineffective in this case (Figure 6.21).

$q(t,(T,\tau))$ and $\rho(t,(T,\tau))$ for $m=1024$ $\qquad$ $q(t,(T,\tau))$ and $\rho(t,(T,\tau))$ for $m=512$

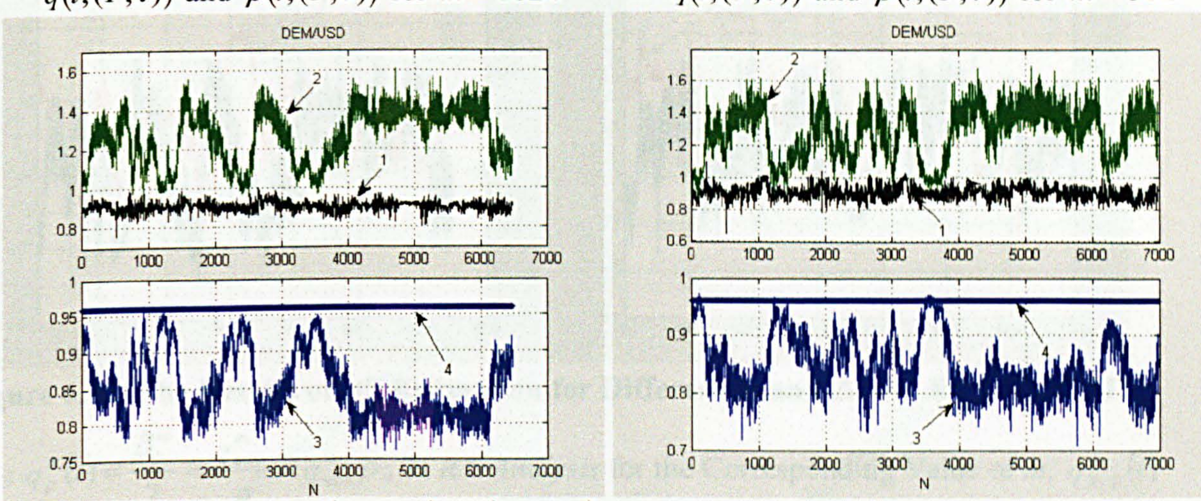


Figure 6.21. Time Series $q(t,(T,\tau))$, $\rho(t,(T,\tau))$ for Different m and $\Delta t=\tau$ for DEM/USD
1: $q_{\min}(t)$; **2:** $q_{\max}(t)$; **3:** The coefficient $\rho(t,(T,\tau))$ of Pairwise (Cross) Correlation of $x=\log(n)$ and $y=\log M_X(n)$; **4:** The Critical Value $\rho_0=0.96$

Also Figure 6.21 indicates that we have another determining parameter available – the coefficient of the pairwise correlation ρ, which not only carries information on the processes occurring, but is also related to the coefficients in the direct and the inverse regressions. The effectiveness of using the OLS method can be substantially increased if instead of the ordinary estimates $q_{\min}=\dfrac{\beta_{\min}}{2}=\dfrac{|a_{0(yx)}|}{2}$ or $q_{\max}=\dfrac{\beta_{\max}}{2}=\dfrac{|a_{0(xy)}|}{2}$ the adjusted estimates $q_\rho=\dfrac{\beta_\rho}{2}=\dfrac{|a_{0(yx)}|}{2\rho}=\dfrac{q_{\min}}{\rho}$ or $q_\rho=\dfrac{\beta_\rho}{2}=|a_{0(xy)}|\dfrac{\rho}{2}=q_{\max}\rho$ are used (Figure 6.22).

But the effectiveness of the realisation of the evaluation algorithms slightly declines because as well as handling the first regression equation, handling the second regression equation and finding of the pairwise (cross) correlation coefficient for the observed parameters is required for further estimation. Direct comparison of the parameters $q_\rho^m(t)$ and $q_{R/S}^m(t)$ is incorrect due to the existence of (generally unknown) time delays τ_ρ and $\tau_{R/S}$ in the sequences. These delays are caused by inertial (accelerative) nature of OLS to changes in values of the reconstruction windows, and by values of OLS estimates. When finding $q_\rho^m(t)$ and $q_{R/S}^m(t)$ there are initially m values in reconstruction windows, but while using OLS estimation, no more than respectively

$m/2$ and m values are used. As a result, staying unknown, the difference $\tau_{R/S} - \tau_\rho > 0$ steadily increases with rising m (Figure 6.22).

$q_\rho^m(t,(T,\tau))$ for $m = 1024$, $q_\rho^m = 1.07$, $q_{R/S}^m = 1.11$ | $q_\rho^m(t,(T,\tau))$ for $m = 512$, $q_\rho^m = 1.08$, $q_{R/S}^m = 1.10$

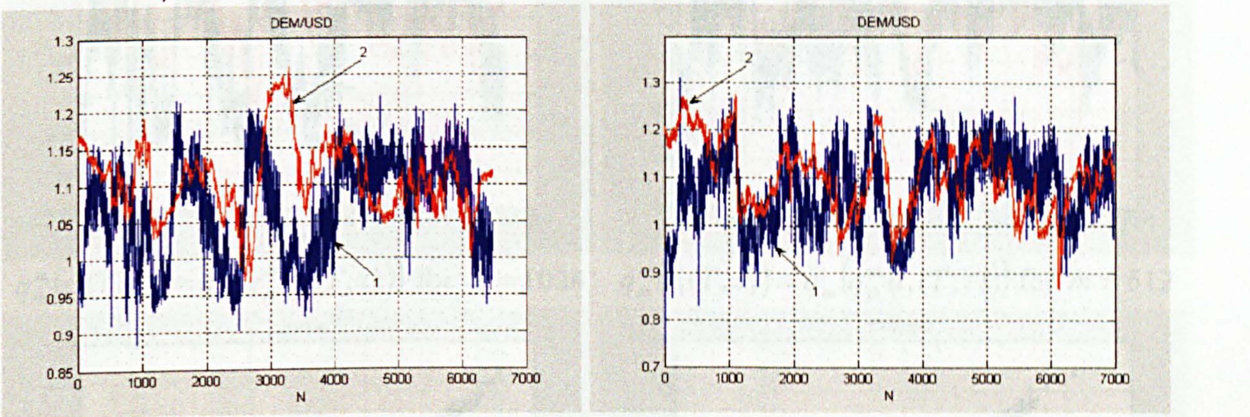


Figure 6.22. The Results of q_ρ^m Estimation for Different *m* and $\Delta t = \tau$ for DEM/USD

1: $q_\rho^m(t) = \dfrac{\beta_\rho^m}{2} = \dfrac{q_{\min}^m}{\rho} = q_{\max}^m \rho$; **2:** *R/S Analysis* for the Corresponding Value of m, $q_{R/S}^m(t)$

For a fixed *m*, for efficient identification the use of parameters $q_\rho^m(t)$ is more preferable, and for comparing sequences $q_\rho^m(t)$ and $q_{R/S}^m(t)$ some mean estimates have to be used. Taking into account the kind of field of dispersion of the observed parameters (Figure 6.18), on the first stage of the analysis, the mean variable $q_m = \dfrac{q_{\min} + q_{\max}}{2}$ has been proposed as an estimation of q. Here, $q_{\min}$ represents the estimates, obtained with equation direct regression; and $q_{\max}$ represents the estimates, obtained with the equation inverse regression. Then $q_m = \dfrac{|k_0|(\rho^2+1)}{2\rho}$, for $\rho \approx 1$, $q_m \approx |k_0|$; and for $\rho \approx 0.7$, $q_m \approx 1.064|k_0|$. That is, since $q_m > |k_0| = q_\rho$ on average the parameters q_m^m give slightly higher estimates in comparison with the parameters q_ρ^m. Experimental checks indicate that direct comparison of $q_\rho^m(t)$ and $q_m^m(t)$ is possible, and with satisfactory accuracy it is also possible to consider that $\tau_\rho = \tau_m$ and $q_\rho^m(t) = q_m^m(t)$ (Figure 6.23).

Mean estimates of q, obtained from the parameters q_ρ^m and q_m^m, can be considered as nearly identical and of similar accuracy; and also satisfactory for fitting the results of the *R/S Analysis* (parameters $q_{R/S}^N$ and $q_{R/S}^m$) (Table 6.11). This result also implies that in any circumstances, the accuracy of mean estimates of q with parameters q_ρ^m and q_m^m can not exceed 1…2%.

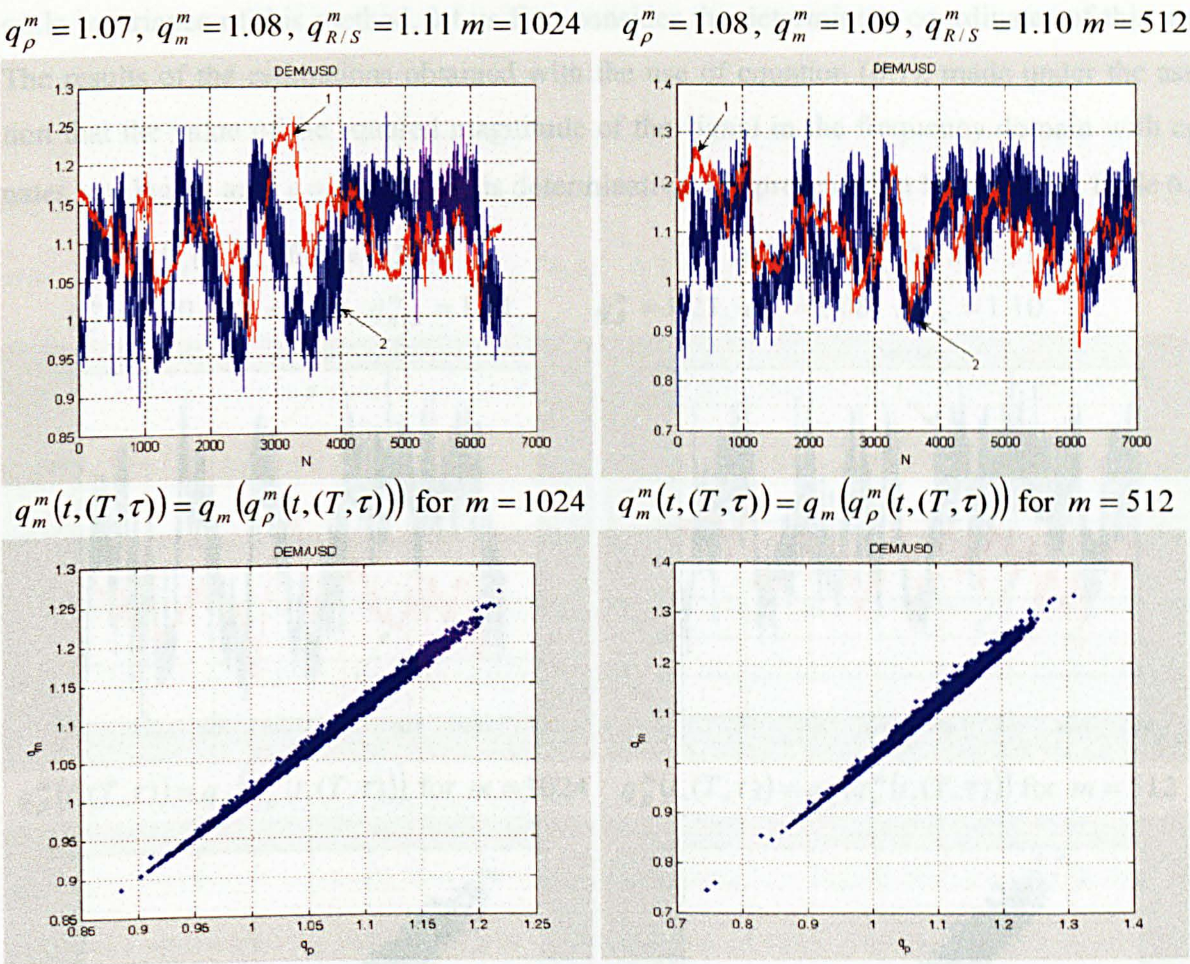


Figure 6.23. The Results of q_m^m Estimation for Different m and $\Delta t = \tau$ for DEM/USD

1: *R/S Analysis* for the Corresponding Value of m, $q_{R/S}^m(t)$; **2:** $q_m^m(t)$

Table 6.11. The Results of the Estimation of the Parameter q_m^m

Data	$q_{R/S}^N$	$m = 1024$			$m = 512$		
		q_β^m	q_m^m	$q_{R/S}^m$	q_β^m	q_m^m	$q_{R/S}^m$
DEM/USD	1.11	1.07	1.08	1.11	1.08	1.09	1.10
JPY/USD	1.11	1.07	1.09	1.11	1.08	1.10	1.12
GBP/USD	1.10	1.08	1.10	1.09	1.08	1.10	1.10
CHF/USD	1.10	1.07	1.09	1.10	1.08	1.10	1.10

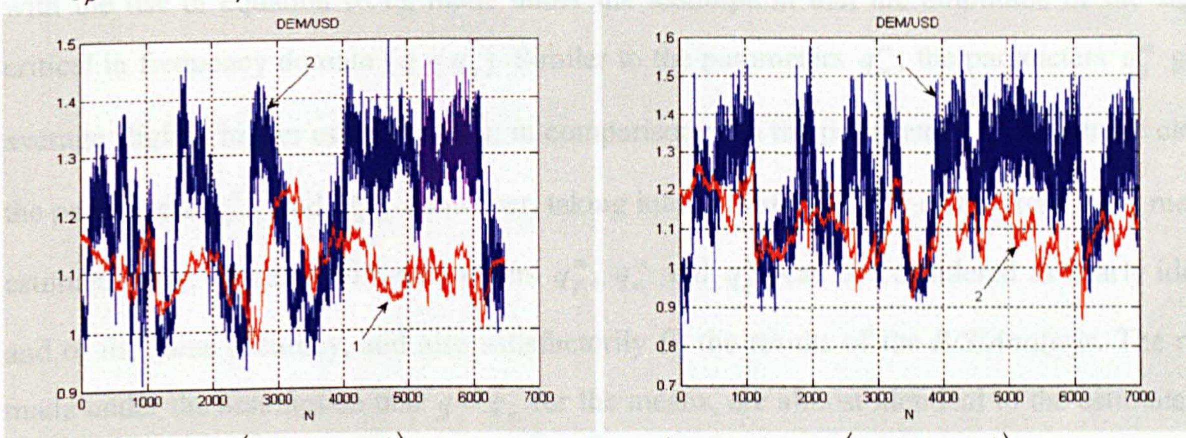
The results obtained suggest that for the estimation of the mean parameter $q = \dfrac{\beta}{2} = \dfrac{|k_0|}{2}$ in problems of identification, the use of the method of orthogonal regression is sensible, and the model (with mean variables $\overline{\rho}$, $\overline{x}$ and $\overline{y}$) is constructed as:

$$\widehat{Y} = \overline{y} + \frac{2\overline{\rho}}{\sigma_0 + \sqrt{\sigma_0^2 + 4\overline{\rho}^2}}(X - \overline{x}), \tag{6.1}$$

$\sigma_0 = \dfrac{\sigma_x}{\sigma_y} - \dfrac{\sigma_y}{\sigma_x}$, and $\overline{\rho} = \dfrac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2 \sum(y-\overline{y})^2}}$. However, taking into account the lack of scale invariance of this method, let us first consider the determining coordinates of this method. The results of the estimations obtained with the use of equation (6.1), made under the assumption that the value of the squared magnitude of the signal in the frequency domain with coordinates $x = \log(n)$ and $y = \log M_X(n)$ is determinative, are presented in Figure 6.24, Table 6.12.

$q_\beta^m(t,(T,\tau))$ for $m = 1024$, $q_\beta^m(t,(T,\tau))$ for $m = 512$,

$q_\beta^m = 1.20$, $q_\rho^m = 1.07$, $q_{R/S}^m = 1.11$ $\quad$ $q_\beta^m = 1.21$, $q_\rho^m = 1.08$, $q_{R/S}^m = 1.10$



$q_\beta^m(t,(T,\tau)) = q_\beta\left(q_\rho^m(t,(T,\tau))\right)$ for $m = 1024$ $\quad$ $q_\beta^m(t,(T,\tau)) = q_\beta\left(q_\rho^m(t,(T,\tau))\right)$ for $m = 512$

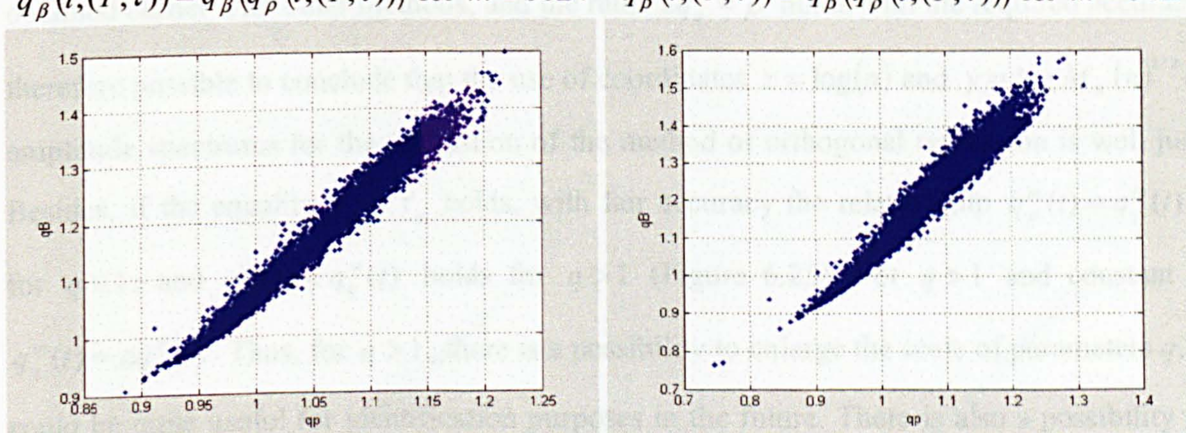


Figure 6.24. The Results of q_β^m Estimation for Different *m* and $\Delta t = \tau$ for DEM/USD

1: *R/S Analysis* for the Corresponding Value of *m*, $q_{R/S}^m(t)$; **2:** $q_\beta^m(t)$

Table 6.12. The Results of the Estimation of the Parameter q_β^m

Data	$q_{R/S}^N$	$m = 1024$			$m = 512$		
		q_ρ^m	q_ρ^m	$q_{R/S}^m$	q_ρ^m	q_ρ^m	$q_{R/S}^m$
DEM/USD	1.11	1.07	1.20	1.11	1.08	1.21	1.10
JPY/USD	1.11	1.07	1.21	1.11	1.08	1.21	1.12
GBP/USD	1.10	1.08	1.22	1.09	1.08	1.23	1.10
CHF/USD	1.10	1.07	1.20	1.10	1.08	1.22	1.10

These results not only indicate the lower accuracy of the results, but also represent contradictions in estimates of the means for the parameter *q*, obtained with the ratio $q = q_\beta = \dfrac{\beta}{2}$ and obtained with earlier methods. This allows for the conclusion of the existence of some unidentified errors in the obtained results, and does not justify the use of the power spectrums (q_β) for a practical realisation of the method of orthogonal regression. Assuming that the determining parameter is the amplitude of the signal in the frequency domain, let us change the coordinates to $x = \log(n)$ and $y = \log M_X(n)^{1/2}$. Table 6.13 and Figure 6.25 present results of the estimation, obtained with the use of equation (6.1), made under the assumption that the amplitude of the signal is critical in frequency domain ($q = q_q$). Similar to the parameters q_m^m, the parameters q_q^m give on average slightly higher estimates of q, in comparison with the parameter q_p^m, which are closer to the parameters $q_{R/S}^N$ and $q_{R/S}^m$. However, taking into account possible errors from these methods, estimates of q, obtained with parameters q_p^m, q_m^m and q_q^m, can be considered as nearly identical and of the same accuracy, and also satisfactorily fit the results of the *R/S Analysis*. The results, made under the assumption that $q = q_q$ for the means, are almost identical to the estimates of q, obtained earlier with other methods, and the ratio $2q_q = \beta$ holds with the required accuracy. It is therefore possible to conclude that the use of coordinates $x = \log(n)$ and $y = \log M_X(n)^{1/2}$ of the amplitude spectrums for the realisation of the method of orthogonal regression is well justified. Besides, if the equality $\tau_p = \tau_q$ holds, with fair accuracy the relationship $q_p^m(t) = q_q^m(t)$ holds for $q \leq 1$; and $q_p^m(t) < q_q^m(t)$ holds for $q > 1$ (Figure 6.25). For $q > 1$ and constant $a > 1$, $q_q^m(t) \approx a q_p^m(t)$. Thus, for $q > 1$, there is a possibility to enlarge the scale of parameters q, which could be quite useful for identification purposes in the future. There is also a possibility to find and eliminate errors. Taking into account when comparing with other methods, the effectiveness of the realisation of the algorithms for the evaluation of $q = q_q$ is high, and all mean estimates, obtained in the frequency domain, are of the same accuracy, we will then use the mean estimate $q = q_q$ as the basic parameter for further analysis.

Table 6.13. The Results of the Estimation of the Parameter q_q^m

Data	$q_{R/S}^N$	$m = 1024$			$m = 512$		
		q_p^m	q_q^m	$q_{R/S}^m$	q_p^m	q_q^m	$q_{R/S}^m$
DEM/USD	1.11	1.07	1.08	1.11	1.08	1.10	1.10
JPY/USD	1.11	1.07	1.09	1.11	1.08	1.10	1.12
GBP/USD	1.10	1.08	1.10	1.09	1.08	1.10	1.10
CHF/USD	1.10	1.07	1.09	1.10	1.08	1.10	1.10

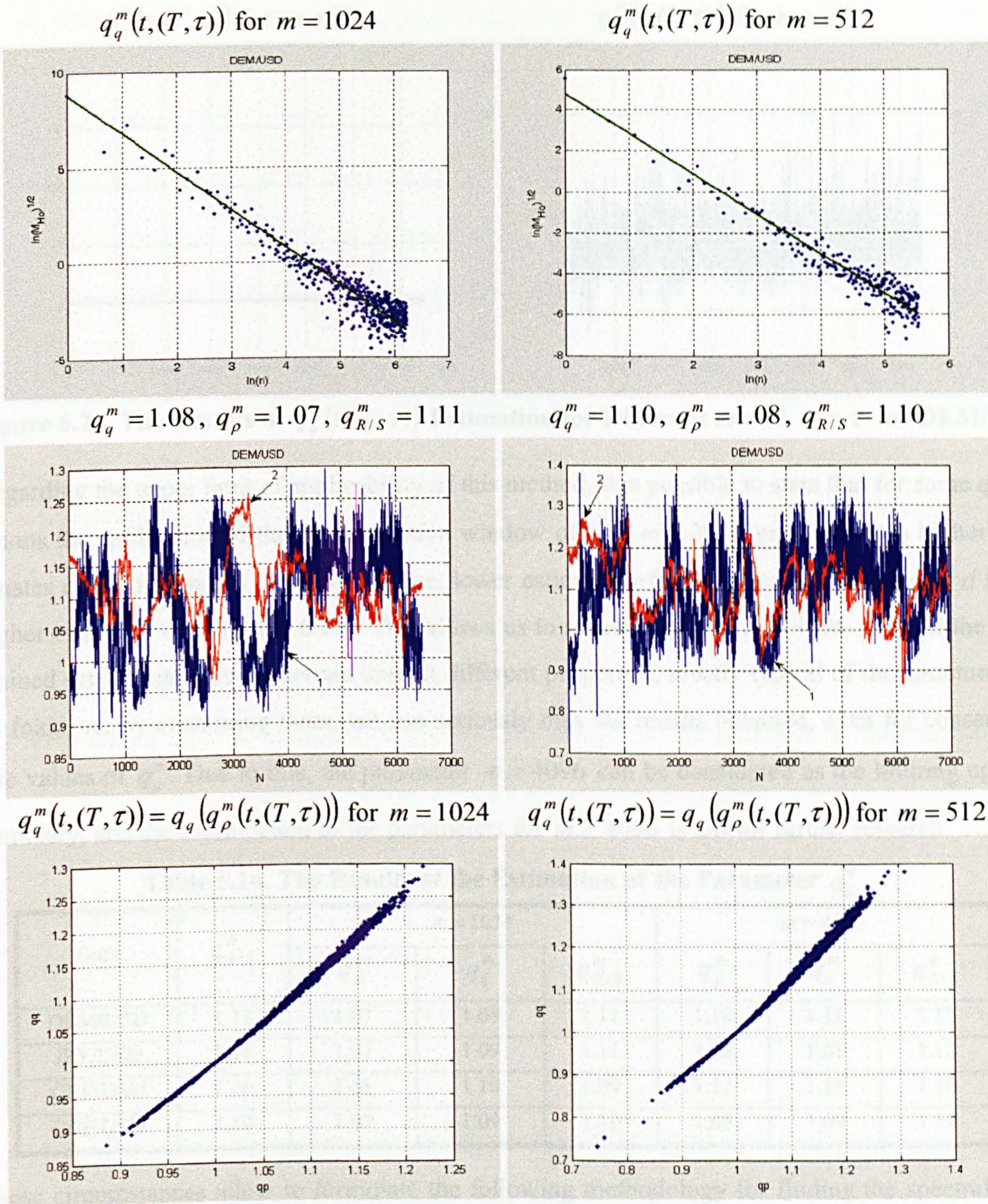


Figure 6.25. The Results of q_q^m Estimation for Different m and $\Delta t = \tau$ for DEM/USD

1: $q_q^m(t)$; **2:** *R/S Analysis* for the Corresponding Value of m, $q_{R/S}^m(t)$

From verification of this method results indicate that for $m \le 64$ and $\tau = 1$ day, regardless of the currency type, starting from $m = 32$ several peaks with magnitude a hundred times higher than the real values of q emerge in the organised series $q(t,(T,\tau))$ (Figure 6.26). For $m = 64$ in the organised time series $q > 2$ only for some quotations there are peaks of high intensity. This justifies the validity of the estimates of a lower limit of applicability for this method, and allows the parameter $m = 64$ to be considered as the limiting lower boundary in this case.

$q_q^m(t,(T,\tau))$ for $m = 32$ $\qquad$ $q_q^m(t,(T,\tau))$ for $m = 64$

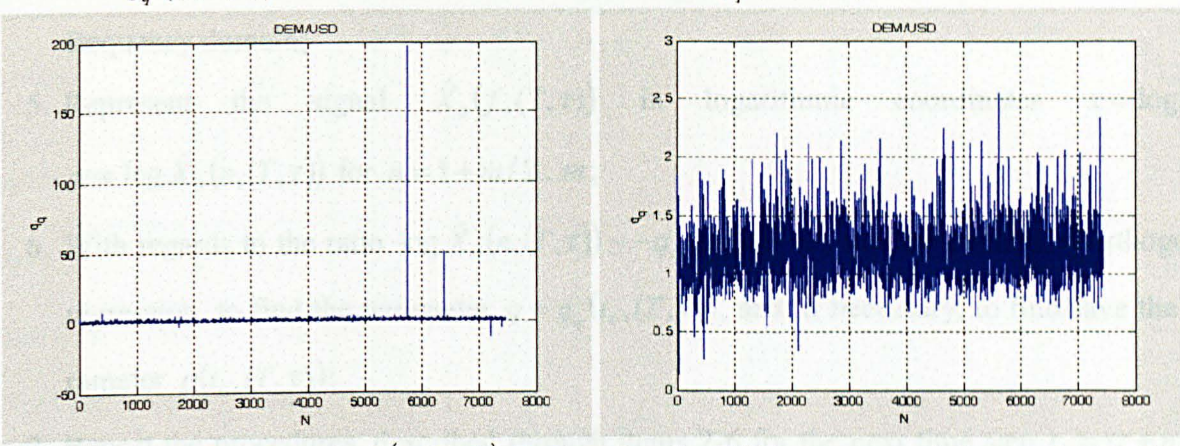


Figure 6.26. The Results of $q_q^m(t,(T,\tau))$ Estimation for Different m and $\Delta t = \tau$ for DEM/USD

Regarding the upper limit of applicability of this method, it is possible to state that for some quotations the application of the reconstruction window of size $m > 2048$ produces both higher estimates of the realisation dimension d (i.e. lower estimates of q), and lower estimates of d (i.e. higher estimates of q) (Table 6.14). This allows us to assume that for $m \geq 4096$ some of the examined sets accumulate (to certain extent) different properties, mostly typical of the structure of its folds, so, by examining them, we can seriously bias the results obtained, even for conservative values of q_q^m. Due to this, the parameter $m = 4096$ can be considered as the limiting upper boundary, and the examination of the parameters for $m \geq 4096$ is left for further research.

Table 6.14. The Results of the Estimation of the Parameter q_q^m

Data	$q_{R/S}^N$	$m = 1024$			$m = 4096$		
		q_p^m	q_q^m	$q_{R/S}^m$	q_p^m	q_q^m	$q_{R/S}^m$
DEM/USD	1.11	1.07	1.08	1.11	1.10	1.12	1.12
JPY/USD	1.11	1.07	1.09	1.11	1.10	1.01	1.12
GBP/USD	1.10	1.08	1.10	1.09	1.12	1.15	1.10
CHF/USD	1.10	1.07	1.09	1.10	1.08	1.09	1.11

These circumstances allow to formulate the following methodology for finding the spectral parameter $q(t,(T,\tau))$ from the experimental data for the purpose of identification of $q_q(t,(T,\tau))$:

1. Take into account the existing constraints (i.e. $2\tau \geq 30$ minutes and $\Delta t = 1$ day), using the theoretical assumptions, to estimate (approximately) the limiting upper and lower boundaries of m. ($64 \leq m \leq 4096$);
2. Choose the minimal boundary size T of the reconstruction window $w = (m-1)\tau$;
3. Find some signal $X_0(t_0,(T,\tau))$, chosen with the reconstruction window from the experimental sequence at the initial moment of time t_0;

4. Using the FFT for the initial signal $X_0(t_0,(T,\tau))$, obtain the signal $\hat{X}_0(f,(T,\tau))$ in the frequency domain;
5. Represent the signal $\hat{X}_0(f,(T,\tau))$ in logarithmic coordinates $x = \log(n)$; $y = \log \hat{X}_0(n,(T,\tau))$ for $n = 1 + m/2 \ldots m$;
6. With regards to the ratio $\log \hat{X}_0(n,(T,\tau)) \sim -q_q \log(n) + \alpha$, use the method of orthogonal regression, to find the parameter $q = q_q(t_0,(T,\tau))$, and, if necessary, to find/save the parameter $\rho(t_0,(T,\tau))$;
7. Repeat the procedures, described through items 2-6 for the next time step $t_1 = t_0 + n_1\Delta t$, where $n_1\Delta t$ is the discreteness of the organised sequence. Thus, for a given T, Δt and τ organise the sequence $q_q^m(t,(T,\tau))$ for the selected values of m, and, if necessary, organise the sequence $\rho(t,(T,\tau))$;
8. Organise the sequence $q_q^m(t,(T,\tau))$ for other values of m, and, if necessary, organise the corresponding sequences $\rho(t,(T,\tau))$;
9. Extract descriptive statistics on the time series $q_q^m(t,(T,\tau))$ obtained; to estimate the uncertainty limits of the parameter q_q^m; and to compare the results obtained with the estimates of the parameter $q^m(t,(T,\tau))$, found with different methods (e.g., using parameters q_ρ^m, q_m^m and $q_{R/S}^m$);
10. Check the results, obtained for boundary values for the size of the reconstruction window (i.e. for $m = 64$ and $m = 4096$), and, if necessary, make corrections to the range of change of m.

6.4 Experimental Estimation of the Mean Values of the Spectral Parameter Including Checking for the Limits of This Method

For a quantitative estimate of the parameter $q = q_q$, obtained with the proposed method, let us first consider the mean values q_q^m, found for the corresponding value of m (Table 6.15). For comparison, Table 6.15 presents the means of $q_{R/S}^N$, calculated for the same period of time under the assumption $q_{R/S} = H + 0.5$, and the corresponding values of q_ρ^m.

Table 6.15. The Results of Finding Mean Values for q_q^m

Data	N	$q_{R/S}^N$	$\frac{q_q^{64}}{q_p^{64}}$	$\frac{q_q^{128}}{q_p^{128}}$	$\frac{q_q^{256}}{q_p^{256}}$	$\frac{q_q^{512}}{q_p^{512}}$	$\frac{q_q^{1024}}{q_p^{1024}}$	$\frac{q_q^{2048}}{q_p^{2048}}$	$\frac{q_q^{4096}}{q_p^{4096}}$
DEM/USD	7495	1.11	$\frac{1.14}{1.10}$	$\frac{1.11}{1.09}$	$\frac{1.10}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.08}{1.07}$	$\frac{1.12}{1.10}$	$\frac{1.12}{1.10}$
JPY/USD	7520	1.11	$\frac{1.11}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.09}{1.07}$	$\frac{1.08}{1.07}$	$\frac{1.01}{1.00}$
GBP/USD	7507	1.10	$\frac{1.13}{1.09}$	$\frac{1.11}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.14}{1.11}$	$\frac{1.15}{1.12}$
CHF/USD	7501	1.10	$\frac{1.13}{1.10}$	$\frac{1.11}{1.08}$	$\frac{1.11}{1.08}$	$\frac{1.10}{1.08}$	$\frac{1.09}{1.07}$	$\frac{1.12}{1.09}$	$\frac{1.09}{1.08}$

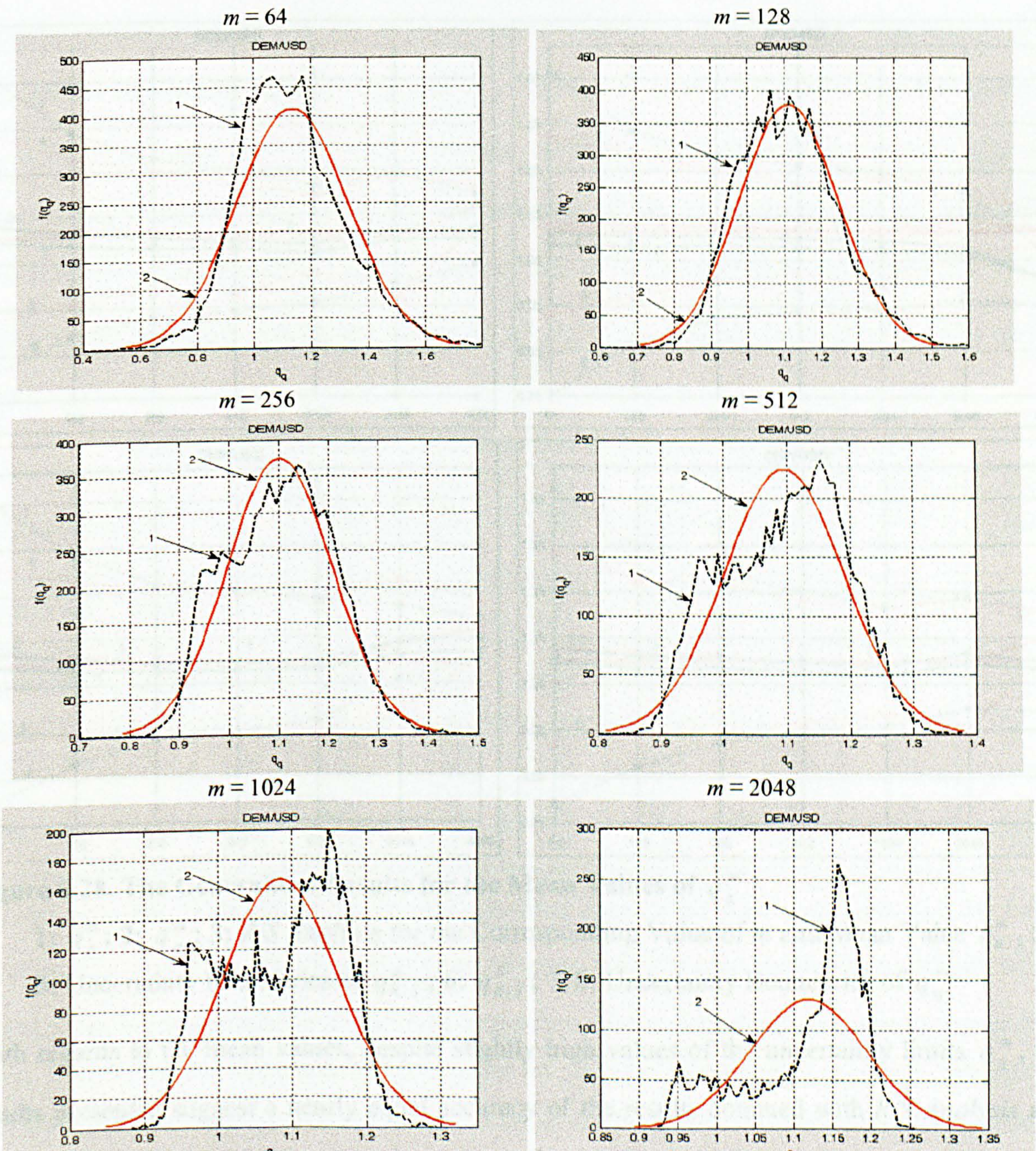
There are two related metrics available: the first – q_q^m – uses modules of increments of the parameters considered; and the second – q_p^m – uses values of their standard deviations. Estimates obtained with metrics q_q^m turn out to be on average closer to the parameters $q_{R/S}^N$ and $q_{R/S}^m$, than estimates obtained with the metrics q_p^m. Table 6.16 presents descriptive statistics for the q_q^m distribution, indicating that the parameters q_q^m can be considered as a homogeneous population because $\hat{s}/q_q^m$ is below 33%.

Table 6.16. Descriptive Statistics on Parameters q_q^m

Data	q_q^m	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$	q_q^m	$\hat{s}$	$\hat{S}_N$	$\hat{K}_N$
m	64				128			
DEM/USD	1.14	0.20	0.83	2.52	1.11	0.14	0.37	0.46
JPY/USD	1.11	0.22	0.06	7.62	1.10	0.14	-0.15	1.97
GBP/USD	1.13	0.20	0.52	3.46	1.11	0.14	0.28	0.66
CHF/USD	1.13	0.21	0.08	3.41	1.11	0.14	0.10	1.17
m	256				512			
DEM/USD	1.10	0.11	-0.04	-0.06	1.10	0.10	-0.01	-0.07
JPY/USD	1.10	0.11	-0.05	0.18	1.10	0.10	-0.29	-0.88
GBP/USD	1.10	0.10	0.19	-0.15	1.10	0.09	-0.16	-0.67
CHF/USD	1.11	0.11	-0.07	-0.34	1.10	0.09	-0.13	-0.64
m	1024				2048			
DEM/USD	1.08	0.08	-0.20	-1.07	1.12	0.07	-0.96	-0.16
JPY/USD	1.09	0.09	-0.22	-1.02	1.08	0.07	-0.29	-1.21
GBP/USD	1.10	0.08	-0.61	-0.53	1.14	0.05	-0.75	0.11
CHF/USD	1.09	0.08	-0.20	-1.03	1.12	0.07	-1.08	0.28
m	4096							
DEM/USD	1.12	0.05	-0.82	-0.13				
JPY/USD	1.01	0.03	0.22	-0.71				
GBP/USD	1.15	0.03	-0.85	0.41				
CHF/USD	1.09	0.07	-0.69	-0.83				

Figure 6.27 presents the distribution of the parameter q_q^m, made from DEM/USD daily quotations. The distributions of the parameter q_q^m are not too oblong around the mode; do not have pronounced heavy tails; and (as a first approximation) can be approximated with a normal distribution with drift (Figure 6.27), which in turn allows us to estimate the uncertainty limits for finding these parameters.

Figure 6.28 presents the results of the estimation of the uncertainty limits for the parameter q_q^m with confidence intervals of no more than 90%, and also presents the generalised results of finding the mean values of q_q^m (according to daily quotations).



$m = 4096$

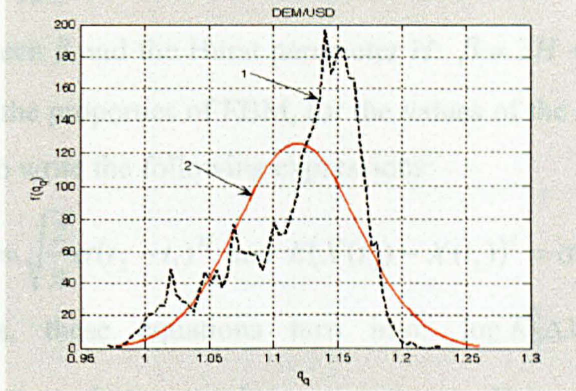


Figure 6.27. Density of Distribution of q_q^m for DEM/USD for Different *m*

1: Density of Distribution of q_q^m; **2:** Approximation with Normal Distribution

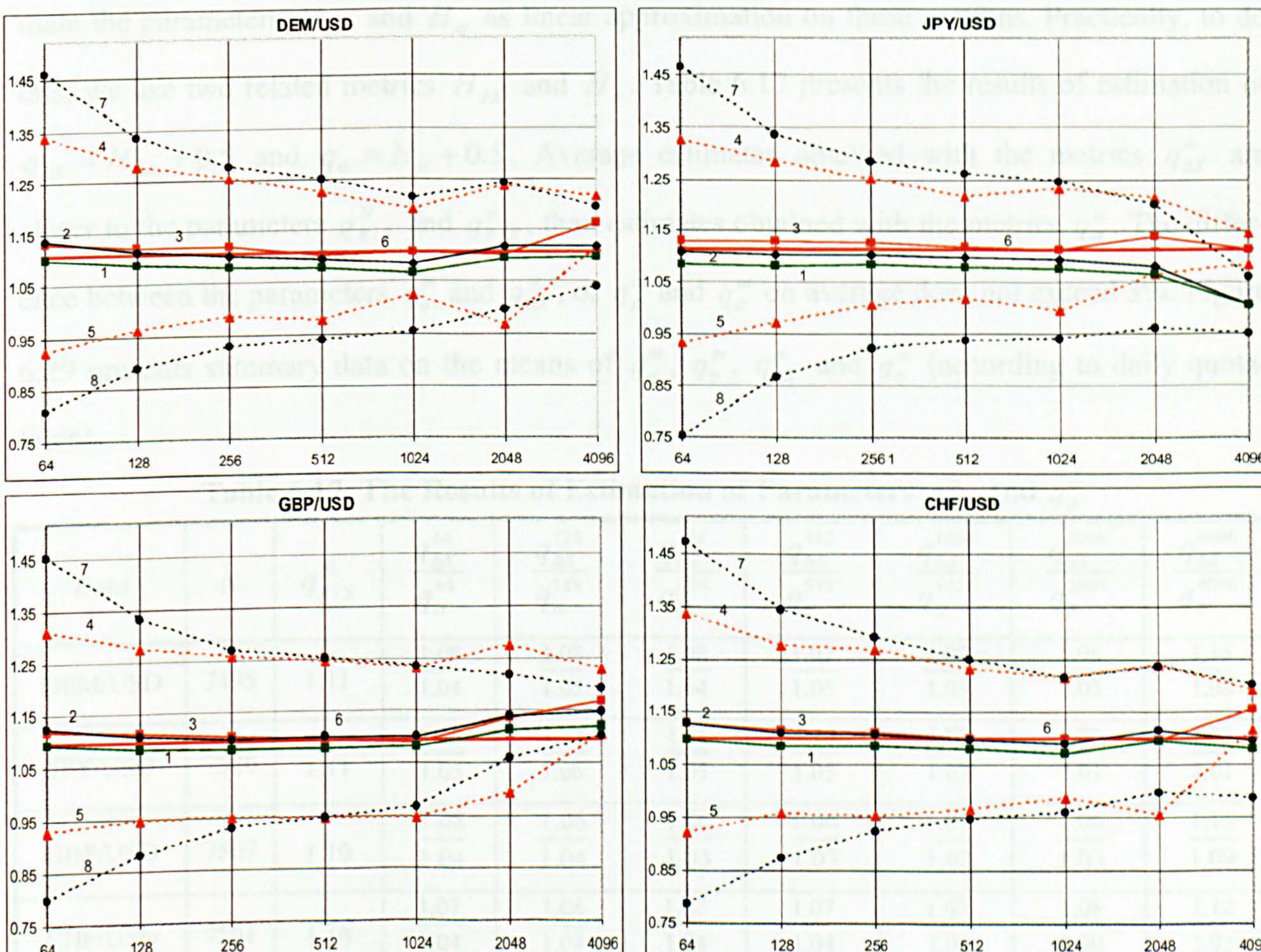


Figure 6.28. The Generalised Results for the Mean Values of q_q^m

1: q_ρ^m; **2:** q_q^m; **3:** *R/S Analysis* for the Corresponding Value of *m* and Mean Value $q_{R/S}^m$; **4, 5:** Uncertainty Boundaries of $q_{R/S}^m$; **6:** $q_{R/S}^N$; **7, 8:** Uncertainty Boundaries of q_q^m

With regards to the mean values, despite slightly high values of the uncertainty limits q_q^m, the results presented suggest a nearly equal accuracy of the results obtained with *R/S Analysis* and the proposed method (if the assumption $q_{R/S} = H + 0.5$ is satisfied). To check the limits of the applicability of the ratio $q_{R/S} = H + 0.5$, let us consider the FBM model, in terms of which $\beta = 2q$ the relation between β and the Hurst parameter H: $\beta = 2H + 1$ has been already examined (Section 4.5). Using the properties of FBM, for the values of the increments and variance of increments it is possible to write the following expressions:

$$E\,|\,X(t_2) - X(t_1)\,| = \sqrt{\frac{2}{\pi}}\sigma(t_2 - t_1)^H \text{ and } E(X(t_2) - X(t_1))^2 = \sigma^2\,|\,t_2 - t_1\,|^{2H}.$$

After taking logarithms, these equations turn into: $\log E[|\Delta X|] = H_{\Delta X} \log|\Delta t| + c_1$ and $\log\sigma(\Delta X) = H_\sigma \log|\Delta t| + c_2$, where c_1 and c_2 are constants. Using the "most line" sections of dependences of $\log E[|\Delta X|]$ from $\log|\Delta t|$, and $\log\sigma(\Delta X)$ from $\log|\Delta t|$, it is possible to estimate the parameters $H_{\Delta X}$ and H_σ as linear approximation on these sections. Practically, to do this, we use two related metrics $H_{\Delta X}$ and H_σ. Table 6.17 presents the results of estimation of $q_{\Delta X} = H_{\Delta X} + 0.5$ and $q_\sigma = H_\sigma + 0.5$. Average estimates obtained with the metrics $q^m_{\Delta X}$ are closer to the parameters $q^N_{R/S}$ and $q^m_{R/S}$, than estimates obtained with the metrics q^m_σ. The difference between the parameters q^m_q and $q^m_{\Delta X}$, or q^m_p and q^m_σ on average does not exceed 3%. Figure 6.29 presents summary data on the means of q^m_q, q^m_p, $q^m_{\Delta X}$ and q^m_σ (according to daily quotations).

Table 6.17. The Results of Estimation of Parameters $q^m_{\Delta X}$ and q^m_σ

Data	N	$q^N_{R/S}$	$q^{64}_{\Delta X} / q^{64}_\sigma$	$q^{128}_{\Delta X} / q^{128}_\sigma$	$q^{256}_{\Delta X} / q^{256}_\sigma$	$q^{512}_{\Delta X} / q^{512}_\sigma$	$q^{1024}_{\Delta X} / q^{1024}_\sigma$	$q^{2048}_{\Delta X} / q^{2048}_\sigma$	$q^{4096}_{\Delta X} / q^{4096}_\sigma$
DEM/USD	7495	1.11	1.08 / 1.04	1.08 / 1.05	1.08 / 1.04	1.07 / 1.05	1.07 / 1.03	1.08 / 1.03	1.15 / 1.08
JPY/USD	7520	1.11	1.10 / 1.05	1.11 / 1.06	1.11 / 1.05	1.10 / 1.05	1.08 / 1.03	1.09 / 1.01	1.08 / 1.01
GBP/USD	7507	1.10	1.08 / 1.04	1.06 / 1.04	1.06 / 1.03	1.06 / 1.03	1.04 / 1.02	1.06 / 1.03	1.15 / 1.09
CHF/USD	7501	1.10	1.07 / 1.04	1.08 / 1.04	1.07 / 1.04	1.07 / 1.04	1.07 / 1.01	1.08 / 1.00	1.12 / 1.05

In relation to the mean values, the results presented suggest the conditions, ensuring the fulfilment of $q_{R/S} = H + 0.5$ with a ratio error of no more than 7%, and justify the possibility of applying these models, established on FBM, regardless of the size of the reconstruction window. Thus, the selected method has to be appropriate not only for identification purposes, but also for objective quantitative estimation of mean values of the fractional dimension, and of the dimensions of the considered time series.

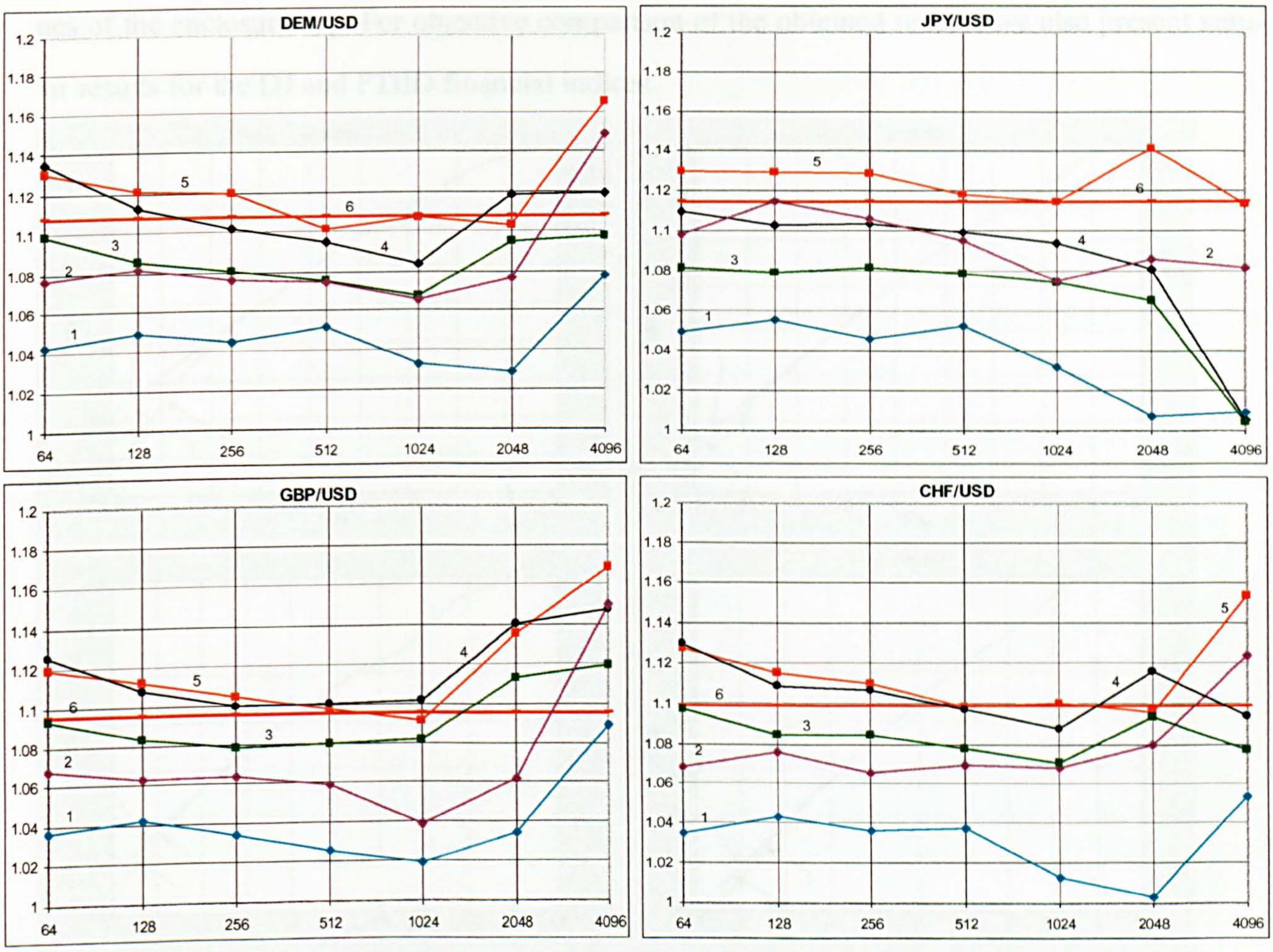


Figure 6.29. The Generalised Results for the Mean Values of $q^m_{\Delta X}$ and q^m_σ

1: q^m_σ; 2: $q^m_{\Delta X}$; 3: q^m_ρ; 4: q^m_q; 5: $q^m_{R/S}$; 6: $q^N_{R/S}$

6.5 Estimation of Stochastic and Chaotic Components in the Structure of the Considered Time Series

It has to be noted that usually $q \neq 1$ and $q > 1$, regardless of the method used for estimation of the spectral parameter. This allows us to consider that in the structure of the time series considered there are not only correlational, but also slowly changing (trend), or even chaotic, components. For an examination of this, let us turn to estimates of the correlation dimension $D_2 = \lim\limits_{\varepsilon \to 0} \dfrac{C(\varepsilon)}{\log \varepsilon}$ of the time series being considered, on the example of DEM/USD and JPY/USD. The correlation integral $C_2(\varepsilon) \equiv C(\varepsilon)$ acts as the main source for the estimation of the dimension D_2. It is only necessary to be aware in advance what type of data is being analysed to be able to interpret the results correctly. Figure 6.30 presents the results of the estimation of the correlation integral for variables H_n for quotations of DEM/USD and JPY/USD and different values of the enclosure m_s. For objective comparison of the obtained results, we also present similar results for the DJ and FTIID financial indices.

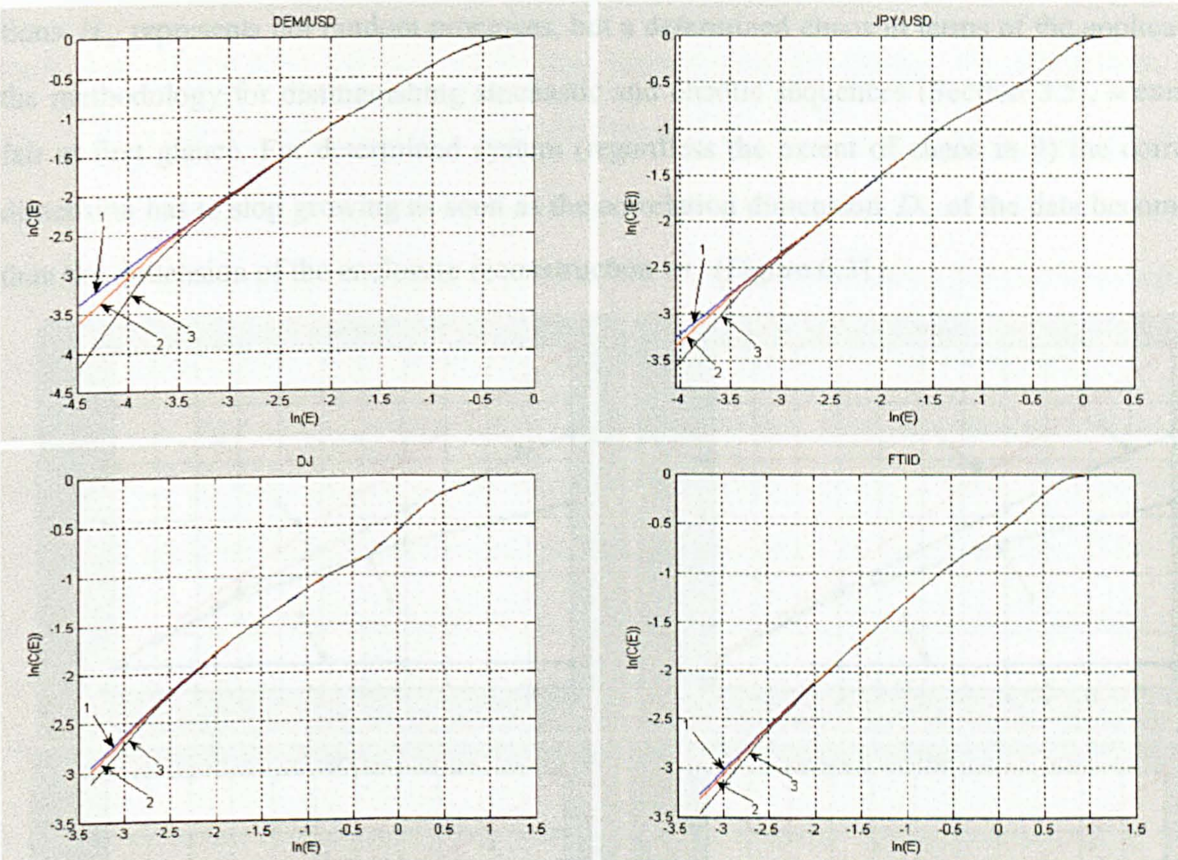


Figure 6.30. Correlation Integral $C(\varepsilon)$ of H_n (for complete time series) for Different m_s

1: $m_s = 1$; 2: $m_s = 4$; 3: $m_s = 12$

Over of the most extended linear segment, the correlation dimensions D_2 of H_n for different values of the enclosure dimension m_s are the following (Table 6.18):

Table 6.18. Correlation dimensions D_2 of H_n for different values of m_s

m_s	**1**	**4**	**8**	**12**
DEM/USD	0.85	0.85	0.86	0.87
JPY/USD	0.83	0.84	0.84	0.85
DJ	0.80	0.82	0.85	0.88
FTIID	0.87	0.89	0.92	0.94

Regardless of the enclosure dimension m_s, the results obtained for D_2 for each of these quotations are nearly the same, but are clearly different from dimension D_2 for a random uncorrelated sequence, where $D_2 \sim m_s$. As a result, we can conclude, that the process of change in quotations H_n is not a random white noise (in the broad sense) or a Gaussian random noise type of process. Thus, finding stochastic and chaotic values, based on the properties of the correlation integral, unambiguously allows us to reject the hypothesis that time series H_n consists of only random uncorrelated components. Based on this, the conclusion that the process of change in the quotations H_n represents not random processes, but a determined chaos in terms of the application of the methodology for distinguishing stochastic and chaotic sequences (Section 3.5), seems quite fair at first glance. For determined system (regardless the extent of chaos in it) the correlation dimension has to stop growing as soon as the correlation dimension D_2 of the data becomes less than the dimension of the enclosure reconstruction m_s (Figure 6.31).

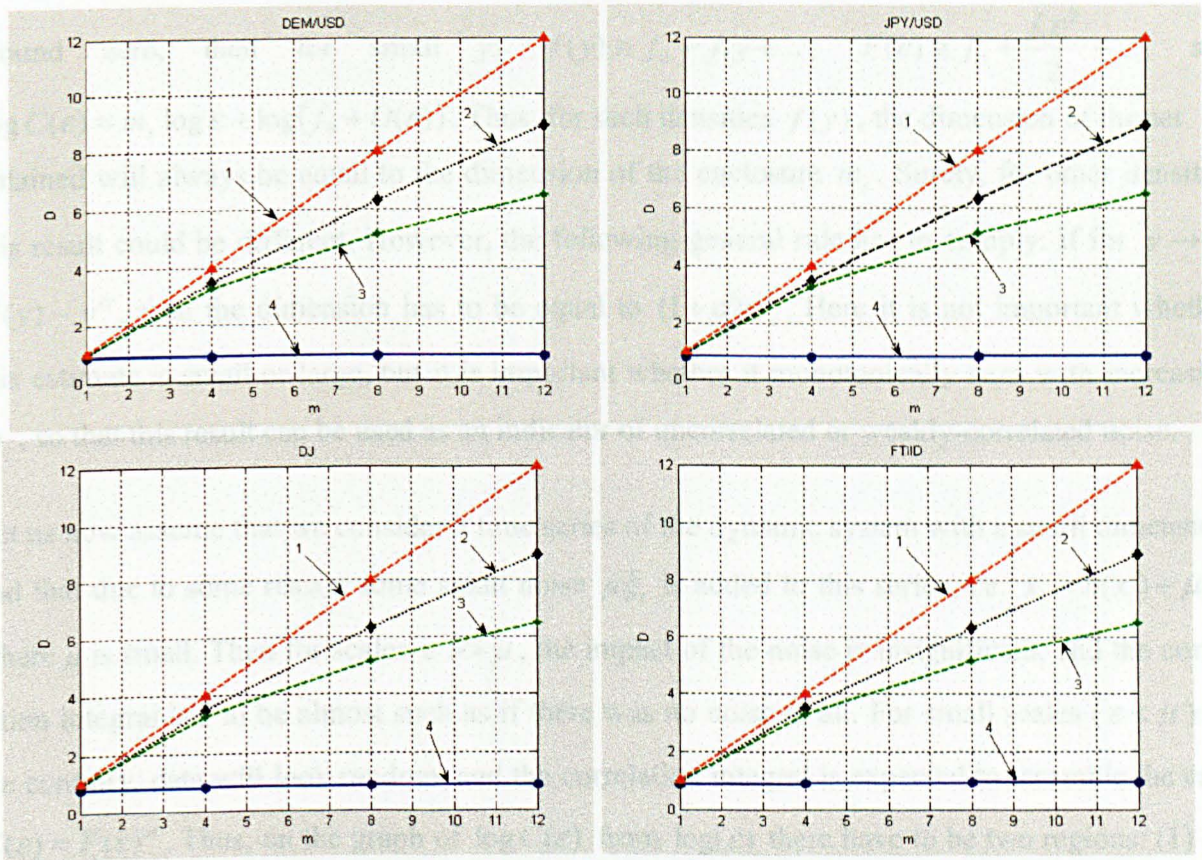


Figure 6.31. Correlation Dimension D_2 of H_n as a Function from the Enclosure m_s

1: Random White Noise Type Process; **2:** Random-Number Generator (MATLAB); **3:** Gaussian Random Noise Generator; **4:** Sequence H_n (for complete time series)

For a comparison Figure 6.31 presents similar results for uncorrelated sequences of different types. However, these results have to be interpreted very cautiously since in this case random but correlated data could be mis-specified as determined. These circumstances suggest considering the problem of the fundamental possibility of a detailed classification of the components of the processes based on the correlation integral. Suppose we analyse the time series of IID values x_i. Basically, we are interested not in the properties of the values x, but in the properties of the values $y_{ij} = |x_i - x_j|$, used for the calculation of differences $\|z_i - z_j\|$. Let random parameters be distributed with density $f(y)$; the distribution function be $F(y) = \int_{-\infty}^{y} f(y)dx$, such that $f(y) = F(y) \equiv 0$ for $y < 0$; and assume that for finding the differences, metric $\|z_i - z_j\|_\infty = \sup_k |z_{ik} - z_{jk}|$ is used. Then the fact that $\|z_i - z_j\|_\infty < \varepsilon$ implies that both $y_{i+k} < \varepsilon$ and $y_{j+k} < \varepsilon$, where $k = 0,...,m_s - 1$. Since the y's are independent parameters, then the probability, i.e. the correlation integral, is $C(\varepsilon) = F(\varepsilon)^m$. If the PDF has no peculiarity (singularity) around zero, then for small y: $f(y) \cong f_0 + f_1 y + ...$; $F(\varepsilon) \cong f_0 + \frac{f_1 \varepsilon^2}{2} + ...$, and $\log C(\varepsilon) = m_s \log \varepsilon + \log(f_0 + O(\varepsilon))$. Thus, for such densities $f(y)$, the dimension of the set obtained will always be equal to the dimension of the enclosure m_s. Surely, for other densities this result could be different. However, the following general rule has to comply: if for $y \to 0$, $f(y) \sim y^\alpha$, then the dimension has to be equal to $(1+\alpha)m_s$. Here it is not important whether this estimate is small or large, but it is important whether it monotonically rises with increasing m_s, so that this result can be used as an indicator of uncorrelated or weakly-correlated noise.

Let us now assume that we consider a time series of the dynamic system with a small dimension, and that due to some reason some small noise $\mu\xi_i$ is added to this series, i.e. $x_i = h(x_i) + \mu\xi_i$, where μ is small. Then for scales $\varepsilon >> \mu$, the impact of the noise is insignificant, and the correlation integral has to be almost such as if there was no noise at all. For small scales ($\varepsilon < \mu$) on the contrary, data will look random, and the correlation integral is expected to resemble the case $C(\varepsilon) = F(\varepsilon)^m$. Thus, on the graph of $\log C(\varepsilon)$ from $\log(\varepsilon)$ there have to be two regions: (1) for the large scales and sufficiently high m_s the slope will stabilise, and (2) for small scales the slope will grow constantly with increasing m_s. According to this behaviour of the correlation integral, it becomes possible to judge on the existence of noise, and to estimate its amplitude. At first glance, while analysing daily quotations of DEM/USD and JPY/USD, exactly these results have been obtained for the correlation integral of the variables H_n (Figure 6.30). It is useful to note that, on the one hand, the FX-market operates under the tight control of the CBs, and the components of the determining processes are not random, but, on the other hand, the actions of a large number of market participants seem to be really random, in the short-term at least. In such cases the correlatedness of noise is essential for justified conclusions. In particular, there are examples [163] of random signals with a spectrum $f^{-\beta}$, where low-frequency parameters dominate the noise and correlations decline very slowly. Also, for these signals there are some intervals of samples N, and some scale segments $[\varepsilon_1, \varepsilon_2]$, on which the correlation integral, obtained for this sample, follows $\log C(\varepsilon) = D \log \varepsilon$. The estimate of the dimension D depends on β, but for sufficiently high m_s is independent of m_s, thus this signal could be easily taken as the determined one. Taking into account the available results, indicating the existence of statistical fractionality in the daily quotations of currency exchange rates, obtained by other researchers, it has to be assumed that we could be faced with this case as well. In support of this assumption, let us present the figures for the empirical autocorrelation function of variables H_n (Figure 6.32):

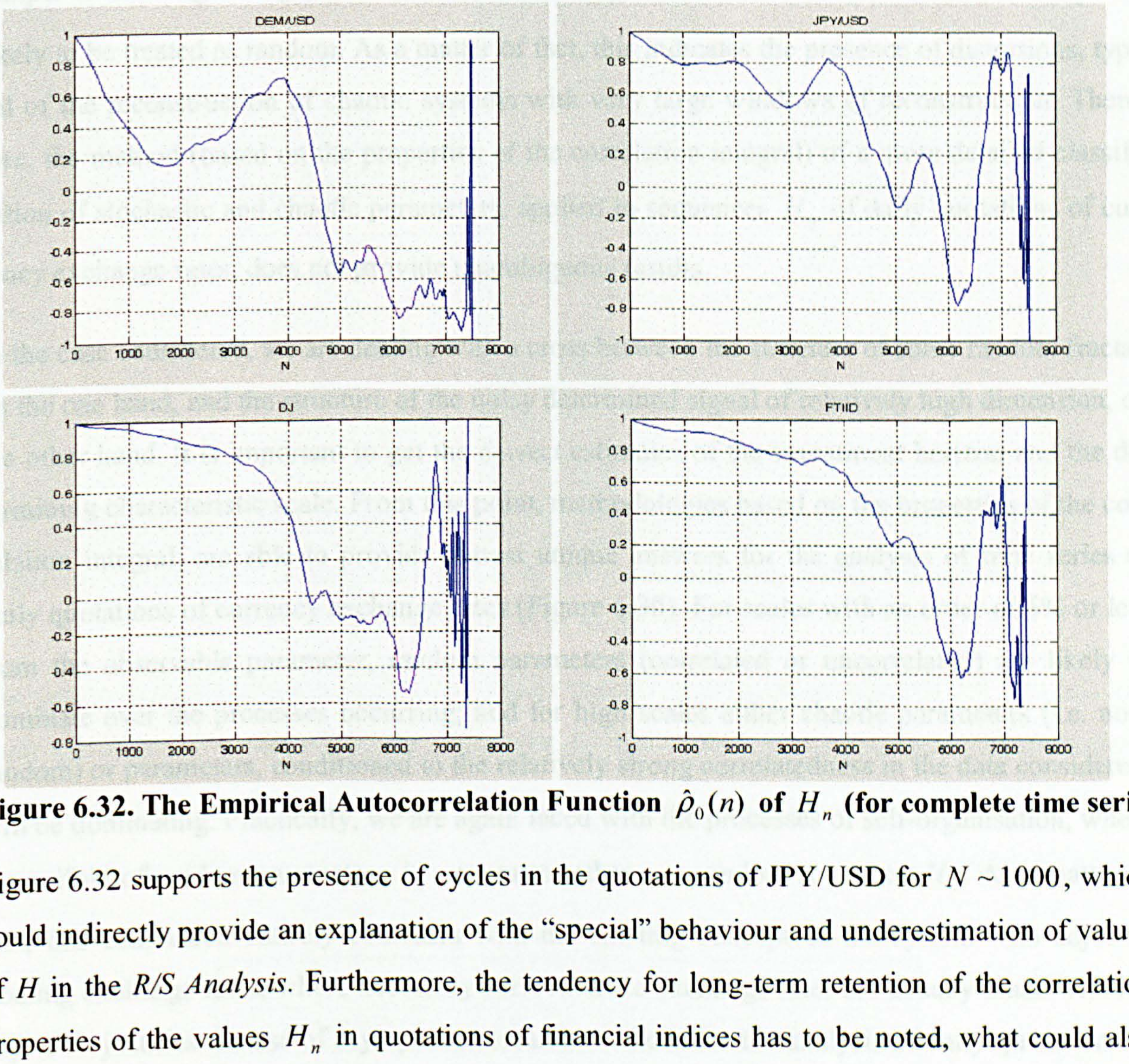


Figure 6.32. The Empirical Autocorrelation Function $\hat{\rho}_0(n)$ of H_n (for complete time series)

Figure 6.32 supports the presence of cycles in the quotations of JPY/USD for $N \sim 1000$, which could indirectly provide an explanation of the "special" behaviour and underestimation of values of H in the *R/S Analysis*. Furthermore, the tendency for long-term retention of the correlation properties of the values H_n in quotations of financial indices has to be noted, what could also indirectly indicate the presence of trend components in its structure. To say on the basis of this data what we are actually dealing with is rather difficult, since the opposite case, when a short sample of data for a chaotic system could be mistreated as a random, is possible.

Indeed, on the graph of the correlation integral it is possible to distinguish two characteristic scales: (1) the maximum distance between $\varepsilon_{\max}$ points, and (2) the scale, from which the graph of the correlation integral becomes almost linear, which makes it feasible to test the fractional characteristics ε_{fr}. It is evident that $C(\varepsilon_{\max}) = 1$, and, following [7], on scales less than ε_{fr}, $\log C(\varepsilon_{fr}) \cong D_2 \log \varepsilon_{fr} - K_2 \tau m_s + c = c_1 - K_2 \tau m_s$, where K_2 is some estimate of system's entropy. Thus, the average slope on the segment $[\varepsilon_{fr}, \varepsilon_{\max}]$ is equal to $\frac{\Delta \log C}{\Delta \log \varepsilon} = \frac{K_2 \tau m_s - c_1}{\log(\varepsilon_{\max}) - \log(\varepsilon_{fr})} = am_s + b$, i.e. it is linearly rising with increasing m_s. If the sample is not long enough, then scales less than ε_{fr} could become unsolvable, and the data is likely to be treated as random. As a matter of fact, this indicates the presence of distortions, typical of the reconstruction of chaotic systems with very large windows of reconstruction. Therefore, the method (based on the properties of the correlation integral) of a more detailed classification of stochastic and chaotic parameters, applied to sequences H_n of daily quotations of currency exchange rates, does not provide unambiguous results.

In the case considered, we are dealing with a cross between the structure of some random fractal, on the one hand, and the structure of the noisy determined signal of relatively high dimension, on the other hand. It is important to get the correct estimates of the investment horizon and the determining characteristic scale. From this point, methodologies based on the properties of the correlation integral, are able to provide almost unique answers for the analysis of time series of daily quotations of currency exchange rates (Figure 6.30). For scales with an order of 5% or less from the observable parameter, random parameters (correlated or uncorrelated) are likely to dominate over the processes occurring; and for high scales either chaotic parameters (i.e. non-random) or parameters, conditioned to the relatively strong correlatedness in the data considered, will be dominating. Practically, we are again faced with the processes of self-organisation, where a manifold of random parameters h_n generates rather non-random sequence H_n. As a matter of fact, this conclusion entirely coincides with the existing concept of the system with adjusted floating exchange rates, where necessary corrections to exchange rates are usually made within a day. This justifies the use of asymptotic research methods for the analysis of daily quotations of the currency exchange rates. The analysis of the sequences H_n of daily quotations of the currency exchange rates can be considered as focused on studying the external macroeconomic factors, caused by reaction to the external environment, rather than by microeconomic factors of the system; and unable to thoroughly analyse cause-and-effect and fractional properties inside the system. For their detailed analysis, the intraday quotations have to be considered. Let us now turn to the analysis of the correlation integral for the time series of logarithmic gain h_n to study the internal factors of the system being considered and the character of its random components (Figure 6.33).

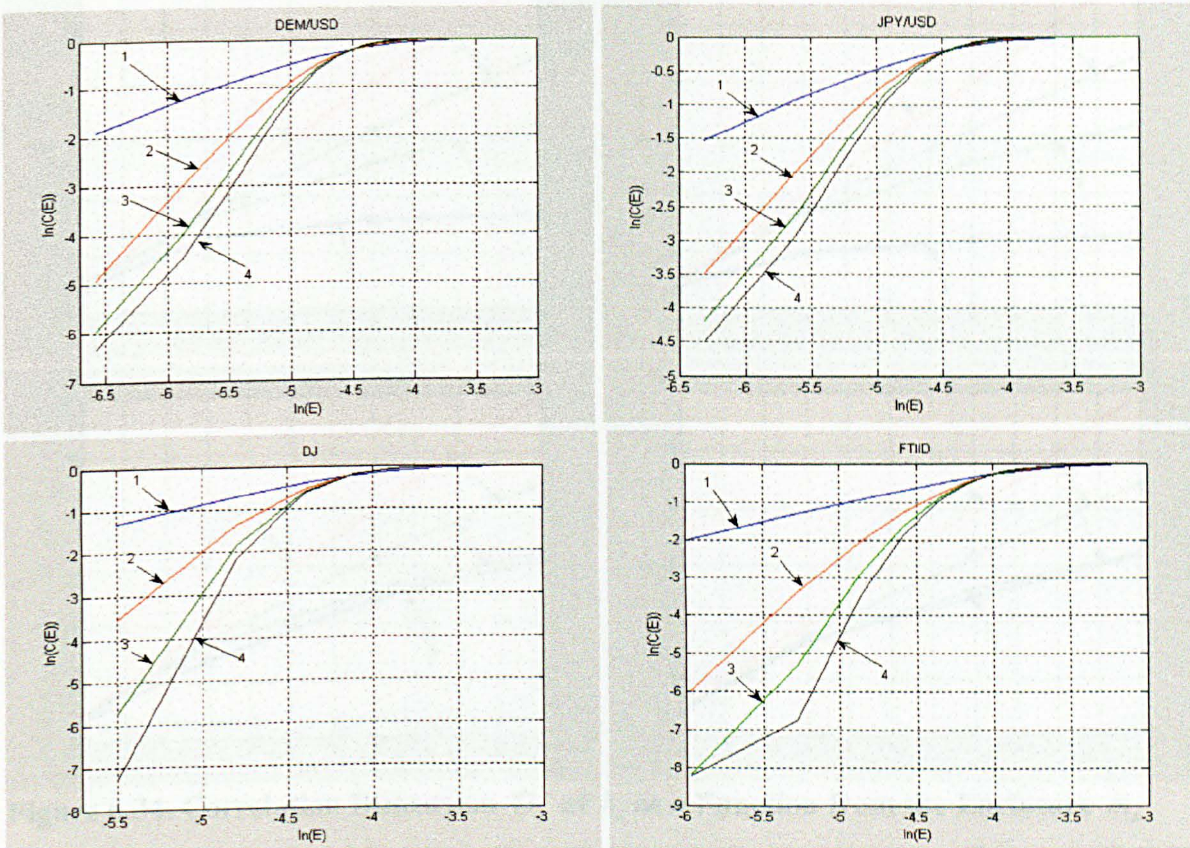


Figure 6.33. Correlation Integral $C(\varepsilon)$ of h_n (for complete time series) for Different m_s
1: $m_s = 1$; 2: $m_s = 4$; 3: $m_s = 8$; 4: $m_s = 12$

The generalisation of the results obtained shows that for scales with an order of 2.5...3% and less from the observable parameter, the sequences h_n has a stochastic rather than chaotic behave (Figures 6.33, 6.34). From the most extended linear segment, the correlation dimensions D_2 of variables h_n for different values of the enclosure dimension m_s are presented in Table 6.19:

Table 6.19. Correlation dimensions D_2 of h_n for different values of m_s

m_s	1	4	8	12
DEM/USD	0.86	2.57	3.12	3.27
JPY/USD	0.77	1.95	2.38	2.55
DJ	0.79	2.78	4.78	5.20
FTIID	0.90	3.26	4.85	5.35

According to the results obtained, for each of these quotations the correlation dimensions D_2 of the variables h_n monotonically increase with rising m_s, which indicates the presence of stochastic uncorrelated or weakly-correlated noise in the processes. Thus, even when using data on daily quotations of currencies, we got one more experimental confirmation of the possibility of the existence of fractional components in the considered sequences.

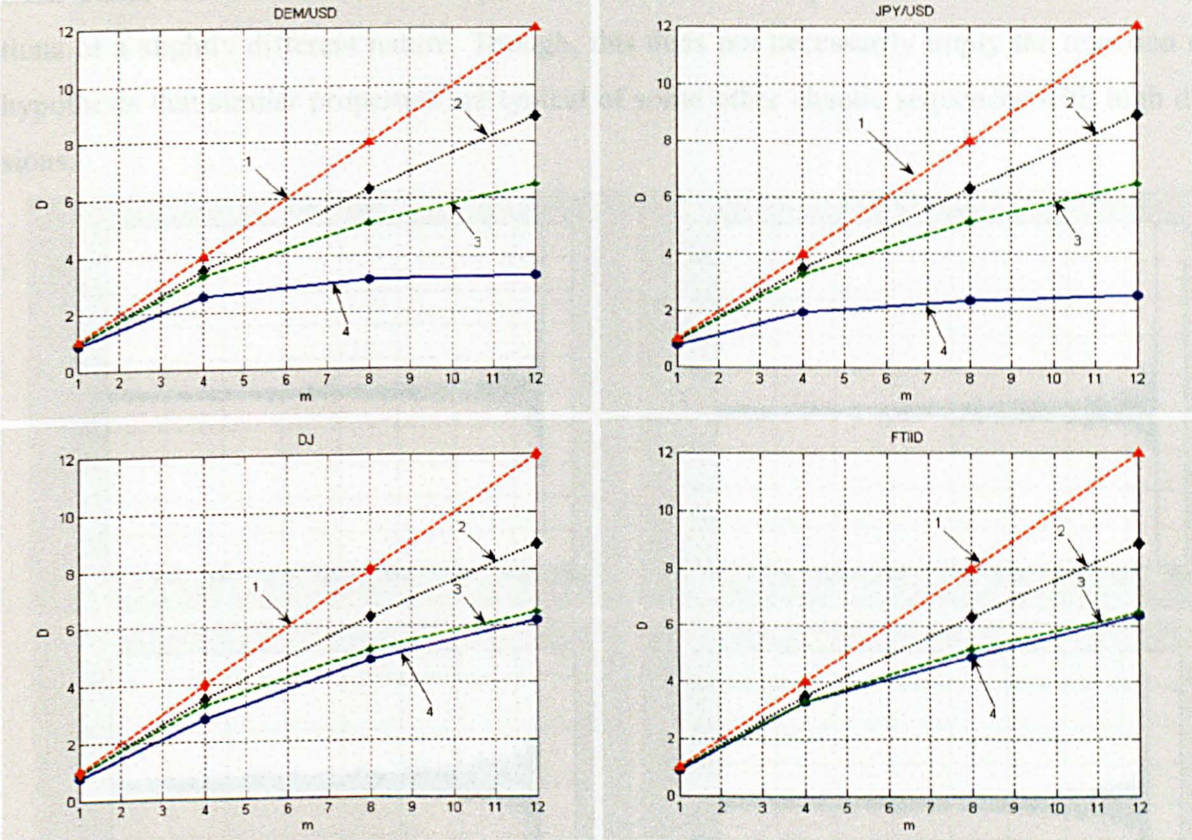


Figure 6.34. Correlation Dimension D_2 of h_n as a Function from the Enclosure m_s

1: Random White Noise Type Process; **2:** Random-Number Generator (MATLAB);
3: Gaussian Random Noise Generator; **4:** Sequence h_n (for complete time series)

The results indicate that there are some differences in behaviour of the dimensions D_2 of h_n between the currency quotations and the financial indices. The dimensions D_2 of the financial indices are very close to the dimensions of the sequences of a Gaussian random noise type (Figure 6.34), which allows us to consider them as uncorrelated or very weakly-correlated sequences. The behaviour of the dimension D_2 of the currency quotations is different from the behaviour of the dimensions D_2 of purely random sequences, which subsequently allows us to suggest that they are more strongly-correlated sequences. It is significant, that the graph of the empirical autocorrelation function $\hat{\rho}_0(n)$ of the variables h_n does not provide any indication of these differences (Figure 6.35). On the one hand, this result indicates that changes in the variables h_n of the currency quotations and of the financial indices are random, while densities of distribution of these parameters and/or spectral parameters of the corresponding quotations are slightly different. For instance, for daily quotations of DJ and FTIID, parameters $q_{R/S}^{N}$ have been 1.04 and 1.06, what is less than for the considered currencies, where $q_{R/S}^{N}$ equal 1.10 and 1.11. On the other hand, this result indicates the possible presence in the processes of more refined correlations of a slightly different nature. Though, this does not necessarily imply the rejection of the hypothesis that similar properties are typical of some other chaotic sequences with high dimensions.

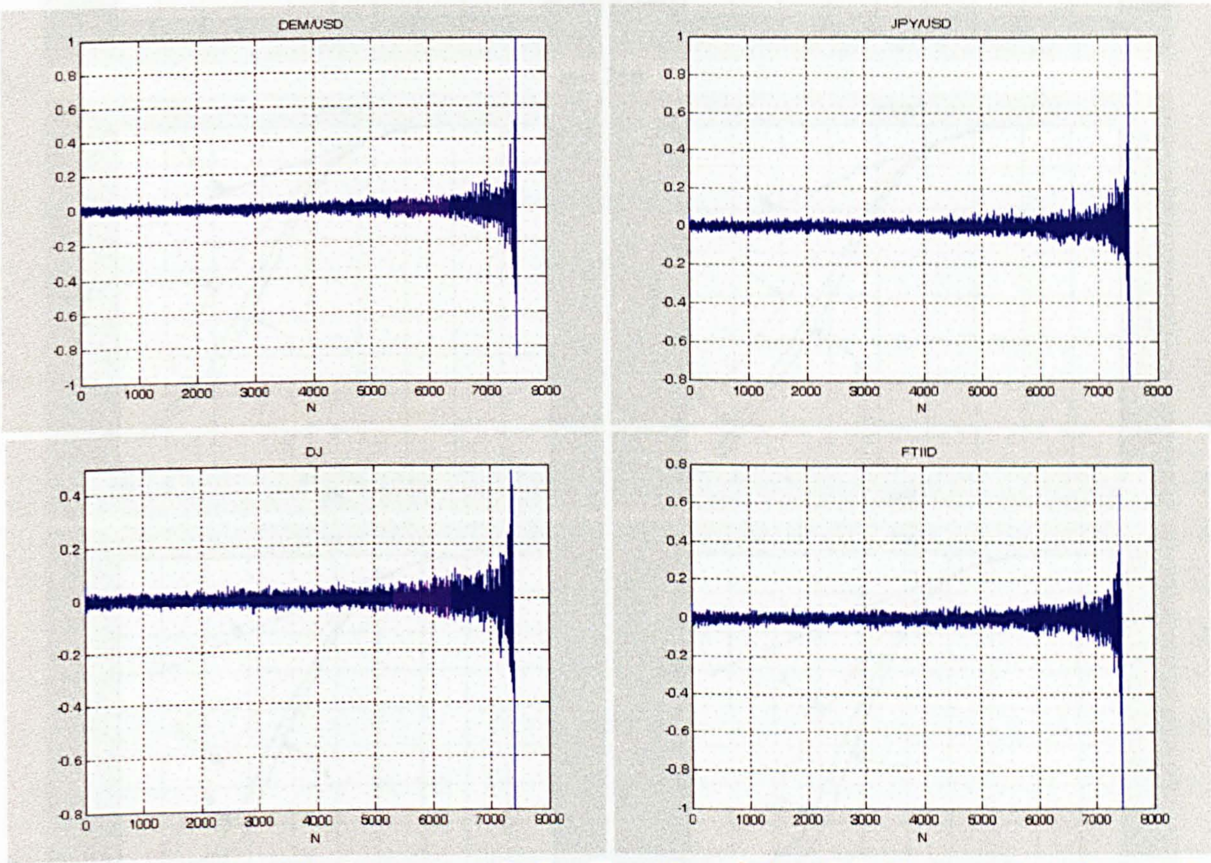
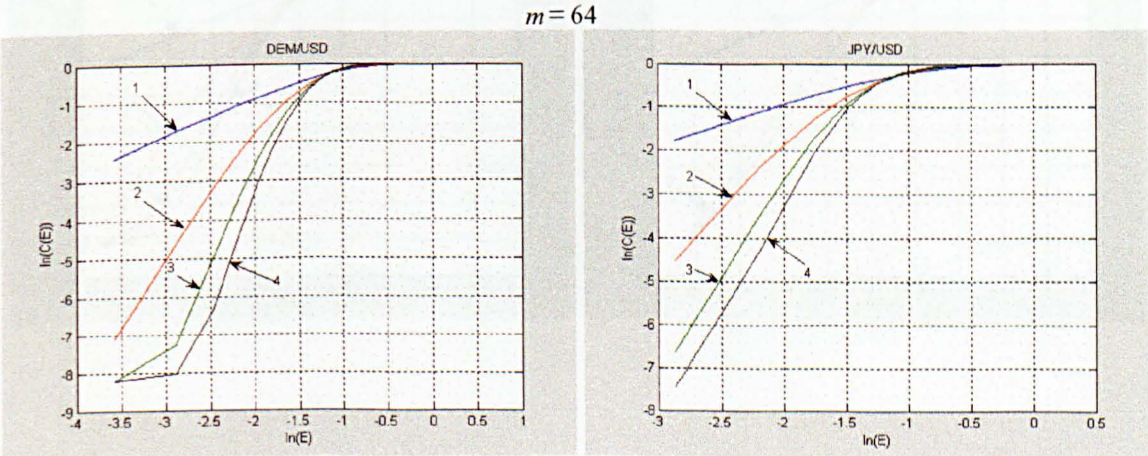
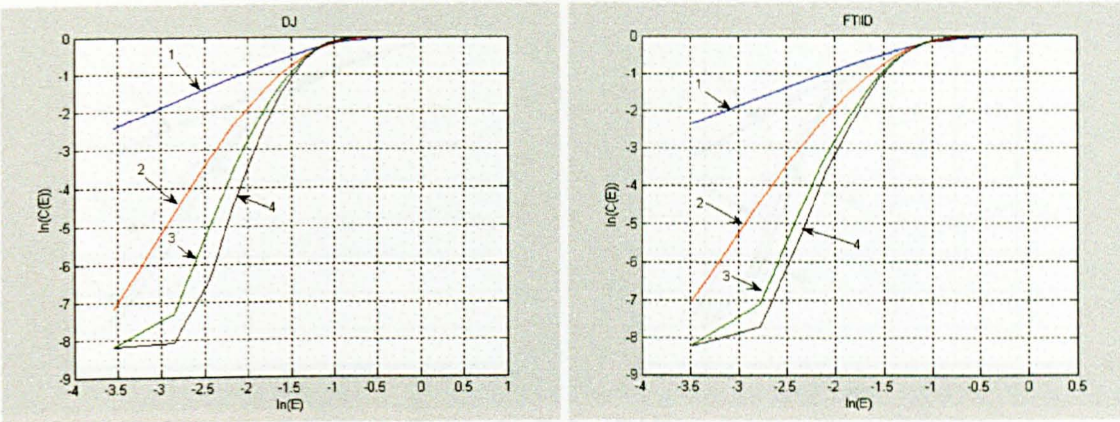


Figure 6.35. The Empirical Autocorrelation Function $\hat{\rho}_0(n)$ of h_n (for complete time series)

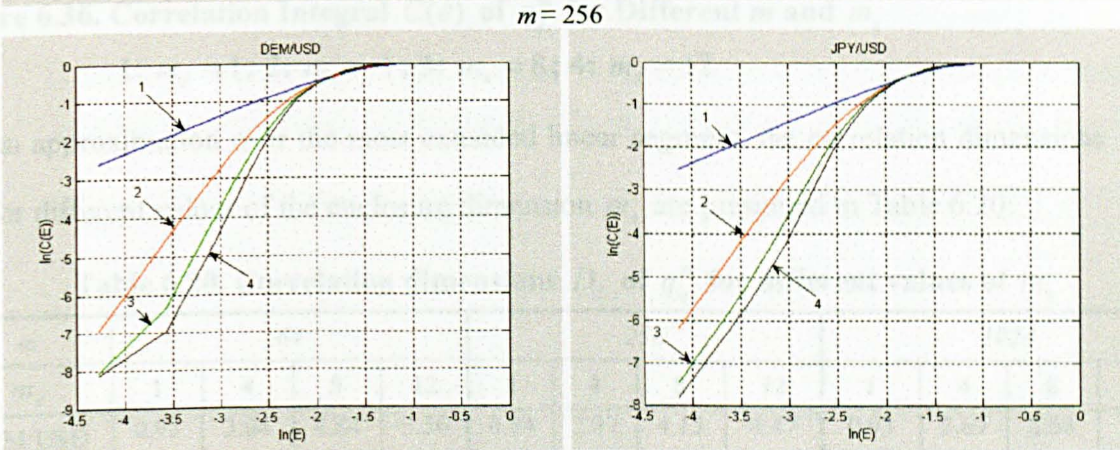
Finally, let us apply this method to determine the nature of the change of the spectral parameter q_q, resulting from a change in the size of the reconstruction window m (Figure 6.36).

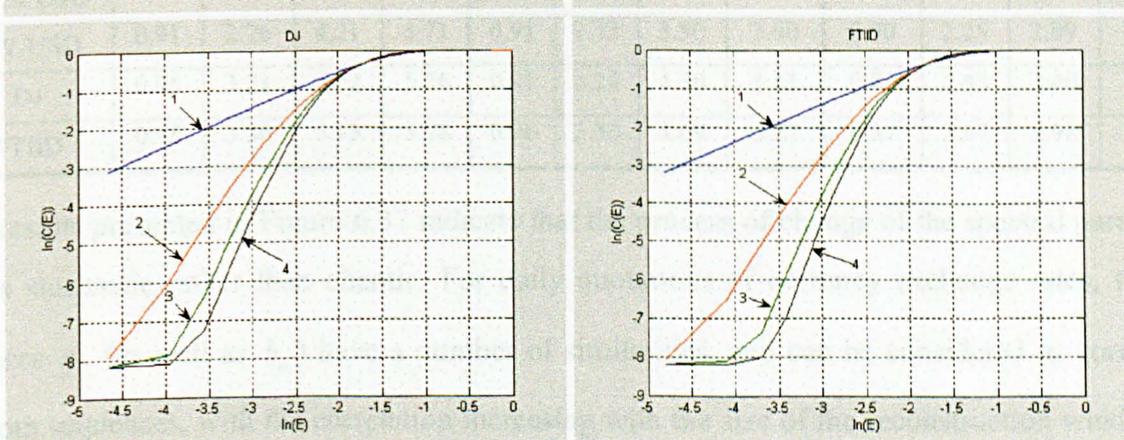
$m = 64$



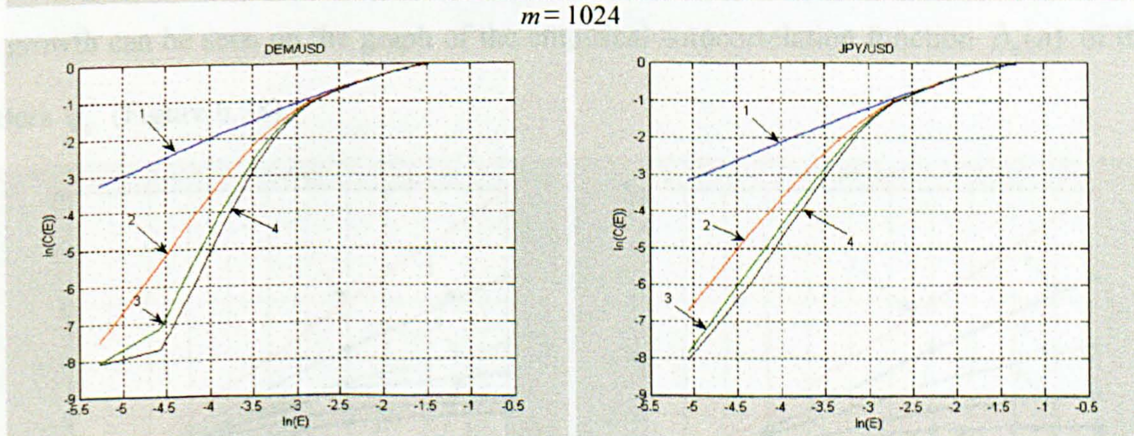

DJ
FTIID
ln(C(E))
ln(E)

$m = 256$


DEM/USD
JPY/USD
ln(C(E))
ln(E)


DJ
FTIID
ln(C(E))
ln(E)

$m = 1024$


DEM/USD
JPY/USD
ln(C(E))
ln(E)

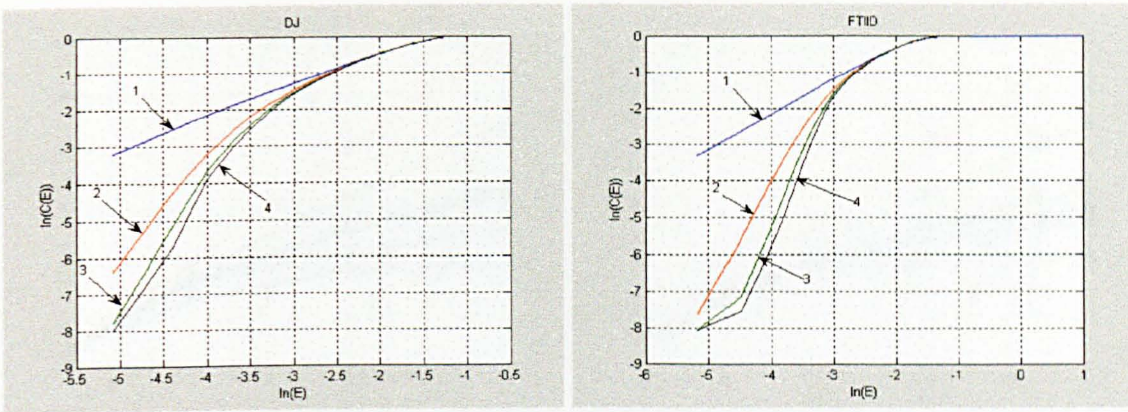


Figure 6.36. Correlation Integral $C(\varepsilon)$ of q_q^m for Different m and m_s

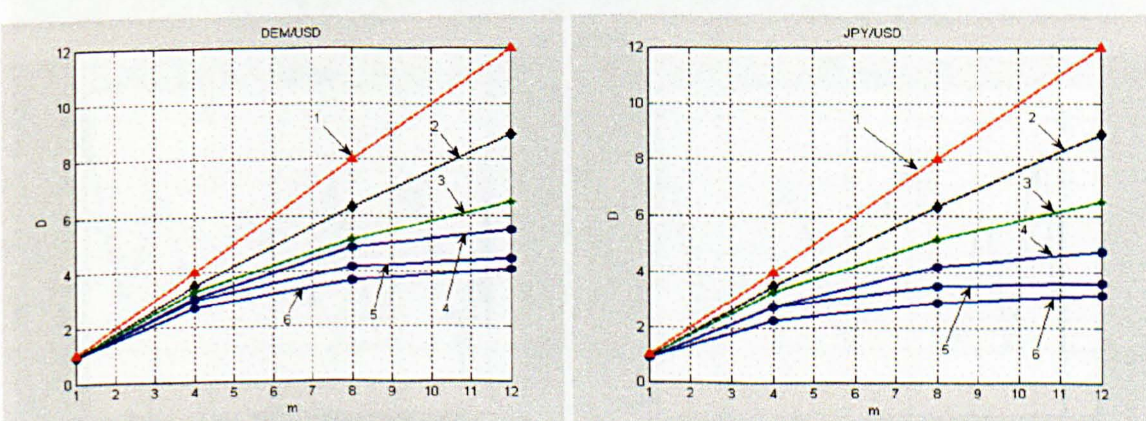
1: $m_s = 1$; 2: $m_s = 4$; 3: $m_s = 8$; 4: $m_s = 12$

For an approximation over the most extended linear segment, the correlation dimensions D_2 of q_q^m for different values of the enclosure dimension m_s are presented in Table 6.20:

Table 6.20. Correlation dimensions D_2 of q_q^m for different values of m_s

m	*64*				*256*				*1024*			
m_s	**1**	**4**	**8**	**12**	**1**	**4**	**8**	**12**	**1**	**4**	**8**	**12**
DEM/USD	0.93	3.04	4.84	5.50	0.94	2.97	4.15	4.47	0.93	2.69	3.68	4.09
JPY/USD	0.91	2.76	4.21	4.71	0.91	2.75	3.50	3.60	0.90	2.25	2.89	3.16
DJ	0.95	3.21	5.12	5.26	0.95	3.24	4.30	5.11	0.87	2.63	3.60	3.74
FTIID	0.96	3.30	5.13	5.78	0.96	3.30	4.69	5.27	0.97	2.97	3.98	4.14

The results presented in Figure 6.37 indicate that the process of change of the spectral parameter q_q is stochastic rather than chaotic. For daily quotations of currency exchange rates, the sequences q_q (as well as h_n) have a number of similarities, and can be considered as correlated random sequences, with the correlation increasing with the size of the reconstruction window m. This growth can be seen on the graph of the empirical autocorrelation function $\hat{\rho}_0(n)$ of the parameters q_q^m (Figure 6.38).



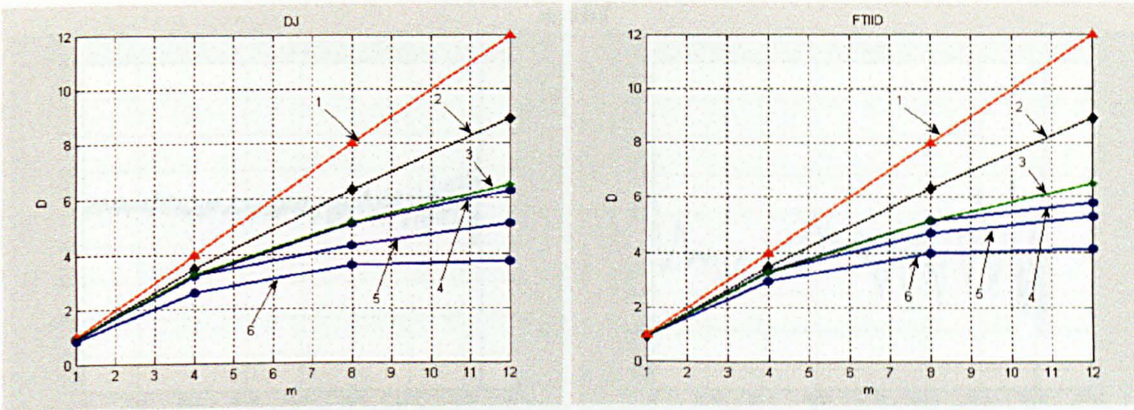
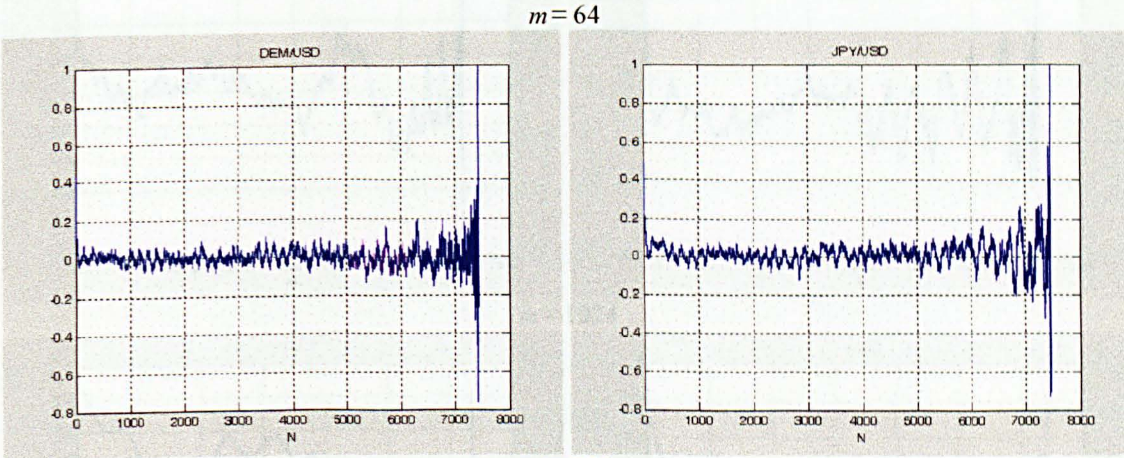


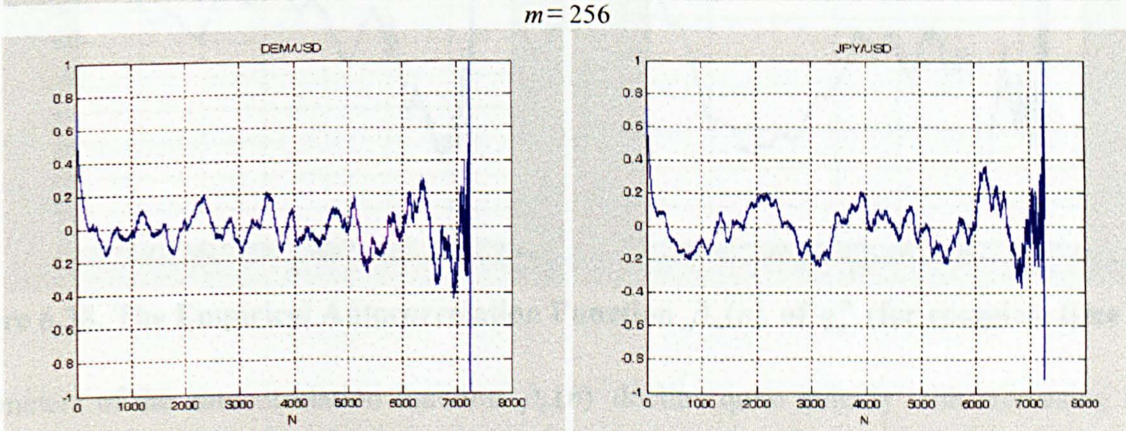
Figure 6.37. Correlation Dimension D_2 of q_q^m as a Function from the Enclosure m_s

1: Random White Noise Type Process; **2:** Random-Number Generator (MATLAB); **3:** Gaussian Random Noise Generator; **4:** q_q^m, $m = 64$; **5:** q_q^m, $m = 256$; **6:** q_q^m, $m = 1024$

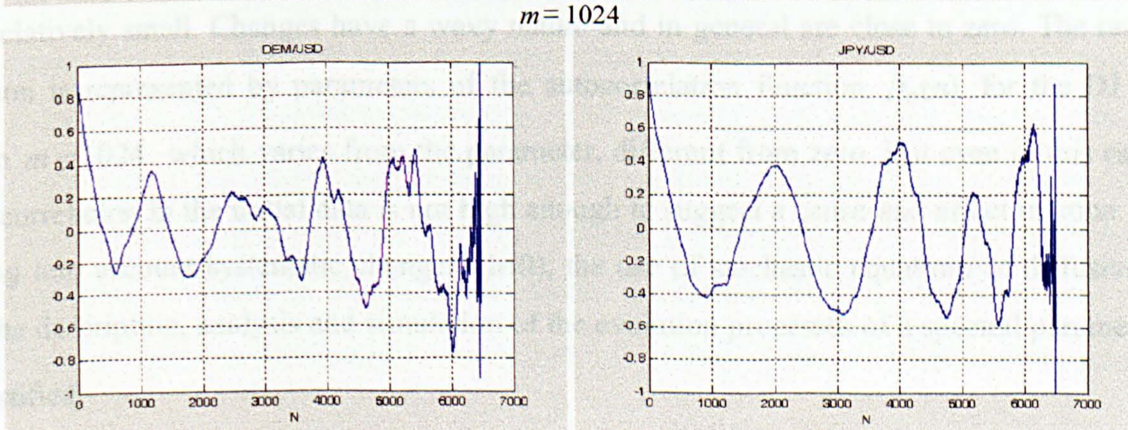
$m = 64$



$m = 256$



$m = 1024$



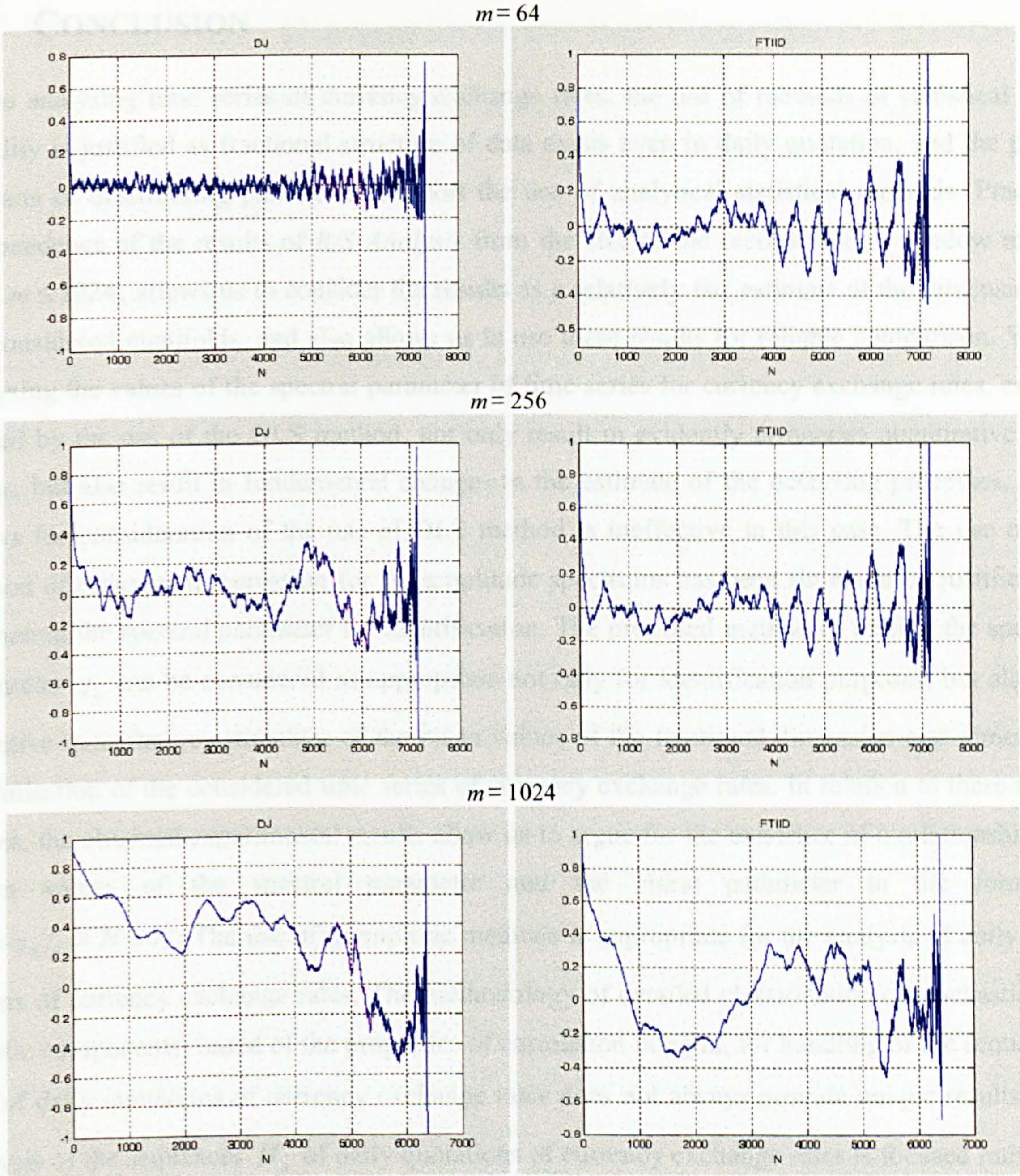


Figure 6.38. The Empirical Autocorrelation Function $\hat{\rho}_0(n)$ of q_q^m (for complete time series)

Parameters of the autocorrelation function $\hat{\rho}_0(n)$ decline quite quickly with increasing n, and are relatively small. Changes have a wavy nature and in general are close to zero. The only exception is represented by parameters of the autocorrelation function $\hat{\rho}_0(n)$ for the DJ index when $m = 1024$, which varies from the parameter, different from zero. But even in this case the intercorrelation in the initial data is not high enough to suggest a cause-and-effect relation. Thus, taking into account systematic changes (drift), the use of stochastic equations of diffusive type for the description, analysis and simulation of the evolution processes of a spectral parameter q_q is justified.

6.6 CONCLUSION

While analysing time series of currency exchange rates, the use of methods of statistical fractionality is justified as fractional structure of data exists even in daily quotation, and the phase portraits of determining parameters support the use of analytical statistical methods. Practical independence of the results of *R/S Analysis* from the size of the reconstruction window m, for $64 \le m \le 1024$, allows us to consider the results as a relatively fair estimate of the dimension of the considered manifolds, and also allows us to use these results for reliable comparison. While analysing the values of the spectral parameter of time series for currency exchange rates, errors, caused by the use of the OLS method, not only result in evidently erroneous quantitative estimates, but also result in fundamental changes in the estimate of the occurring processes, what allows for consideration of the use of OLS method is ineffective in this case. The use of the method of orthogonal regression for the amplitude spectrums has been shown to be justified for estimating the spectral parameter for identification. The proposed method of finding the spectral parameter q_q can be considered as appropriate not only for identification purposes, but also for objective quantitative estimation of the mean values of the fractional dimension and dimension of realisation of the considered time series of currency exchange rates. In relation to these mean values, the obtained experimental results allow us to argue for the existence of a relationship between values of the spectral parameter and the Hurst parameter in the form of $q_q = q_{R/S} = H + ½$. The use of asymptotic methods is appropriate for the analysis of daily quotations of currency exchange rates. The methodology of detailed classification of stochastic and chaotic components, based of the properties of correlation integral, for handling of the sequences H_n of daily quotations of currency exchange rates does not always provide unique results. The analysis of the sequences H_n of daily quotations of currency exchange rates is focused rather on studying the external macroeconomic factors, caused by reaction to the impact of the external environment, than by internal microeconomic factors of the system, and is therefore unable to thoroughly analyse cause-and-effect relationships and fractional properties occurring inside of the system. For their detailed analysis, either intraday quotations, or other type of sequences have to be considered. Changes in parameters h_n of currency quotations and financial indices are (possibly) random, but density distributions of these parameters and/or of the spectral parameters of the corresponding quotations are slightly different. For describing, analysing and simulation of the evolution processes of the spectral parameter q_q, the use of diffusive-type stochastic equations, accounting the existence of systematic changes (drifts), can be justified, at first approximation.

CHAPTER VII. Discussion on the Results of Theoretical and Experimental Estimation

7.1 Introduction

This chapter determines time series $q_q^m(t)$ for the considered currencies using the proposed methodology. It shows that for identification purposes, concerning the time series $q_q^m(t)$, it is possible to use the averaged time series $q_q^{\Sigma}(t)$. The results support the retention of some important properties of initial quotations in the sequences $q_q^m(t)$ and $q_q^{\Sigma}(t)$. The possibility of the existence of many local equilibria of the FX-market in the general case is supported and the use of the coefficient of pairwise correlation $\rho^m(t)$ of the direct and the inverse regressions of the spectral parameter $q_q^m(t)$ is proposed to find regions which determine the behaviour of the system. In terms of the theoretical model, this chapter presents a methodology for experimentally finding parameters, that determine systematic and fluctuating components of the process of the evolution of the spectral parameter. The existence of a statistical relationship between these components has been supported. Based on asymptotic methods of analysis, qualitative estimates of the key parameters of the process of the evolution of the spectral parameter, including complex indices for the whole process, are provided. The dominating impact of empirical volatility of the process of the evolution of the spectral parameter, whose distribution function for the complete sequence is far from normally distributed, is supported. Also results support the fact, that for identification and classification purposes, to a first approximation, the process of the evolution of the spectral parameter can be described with fair accuracy via diffusive types of equations, including the FPK equation with variable parameters.

7.2 Finding the Specificity of Practical Application of the Proposed Method

Let us apply the proposed methodology for finding parameters $q_q^m(t)$. Figure 7.1 presents the results of finding of parameters $q_q^m(t)$ for $m = 128,...,2048$, shifted to the beginning of the reconstruction window. In comparison with the results represented in Table 6.1, Figure 7.1 indicates a shift of 2048 points.

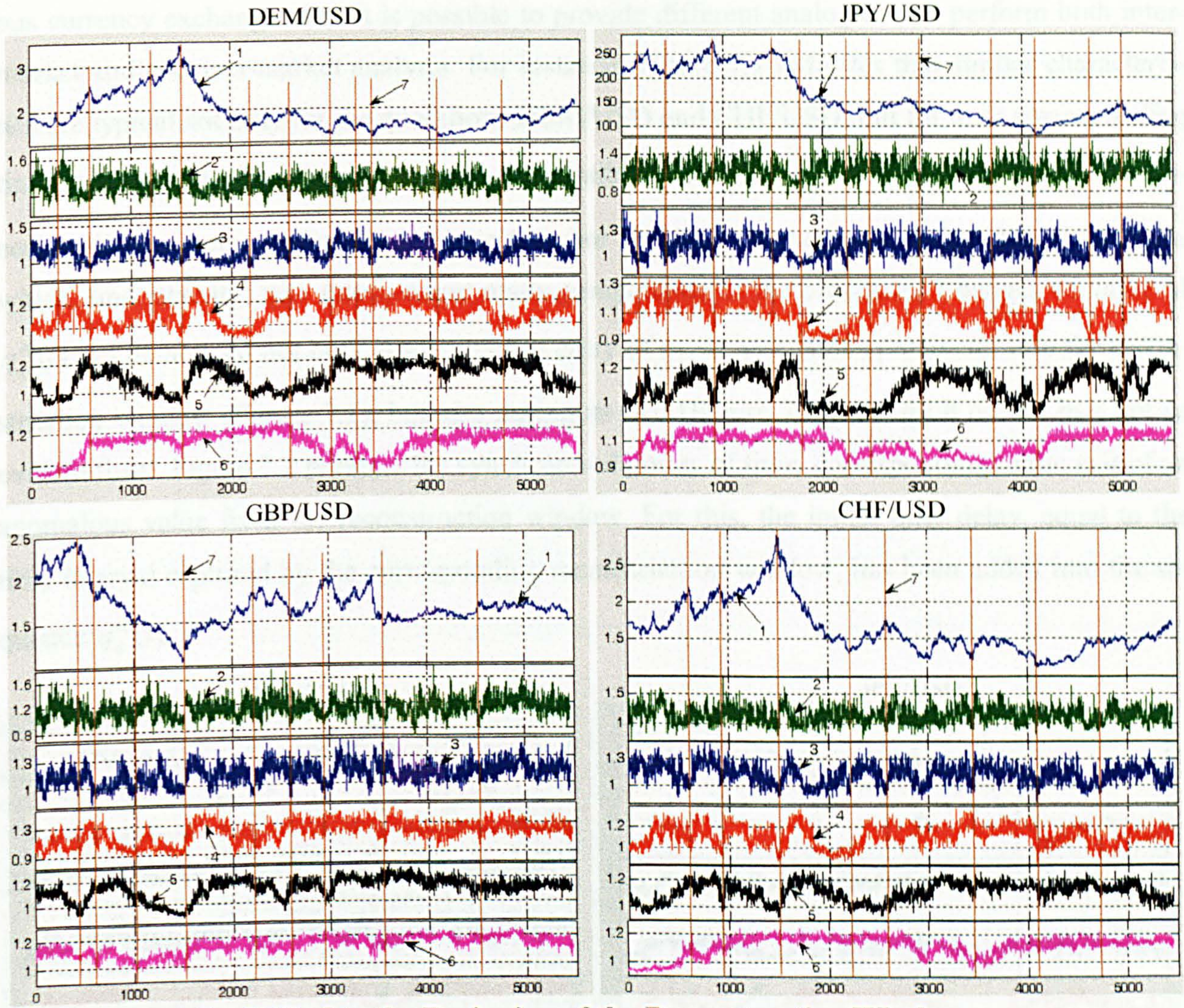


Figure 7.1. $q_q^m(t)$, **Shifted to the Beginning of the Reconstruction Window**

1: initial quotation; **2:** $m = 128$; **3:** $m = 256$; **4:** $m = 512$; **5:** $m = 1024$;

6: $m = 2048$; **7:** synchronised points of time

The obtained results not only allow for general consideration of quotations' behaviour, but, with regards to the size of selected reconstruction window, allow also for analysing data at any point of time, both in terms of "persistence" and "anti-persistence", and in terms of the existence/absence of after-effects (memory effects). For the points of time when $q_q^m(t) \approx q_q^m$ the existing market tendencies usually remain. Changes in the existing tendencies, i.e. the emergence of anomalous values, are typical for the points of local extremums (for relatively sharp changes of parameters $q_q^m(t)$) where $q_q^m(t) \to q \sim 1$ and where $q_q^m(t) > q_q^m$. It is significant that the time series $q(t)$, that is obtained using direct regression, does not have these properties (Figure 6.16). This behaviour directly indicates the possibility of using the time series $q_q^m(t)$ for identification of behaviour of the system, and finding anomalous values. When comparing quotations of various currency exchange rates it is possible to provide different analogies and perform both inter-market and between-market analysis. For instance, Figure 7.1 indicates that similar characteristics are typical not only for the quotations DEM/USD and CHF/USD, but for their corresponding time series $q_q^m(t)$. These results support the possibility of using $q_q^m(t)$ as an independent macro-economic parameter, especially because they are obtained on the basis of previous quotations, which, undoubtedly, take into account many components of the processes. Changes of value of $q_q^m(t)$ can be noticed not only at the point of entry of some anomalous parameter into the reconstruction window (Figure 7.1), but also during its exit (Figure 7.2), though it occurs in a not so evident form. Figure 7.2 presents the synchronised points of time, corresponding to the exit of an anomalous value from the reconstruction window. For this, the initial time delay, equal to the time interval captured by the corresponding reconstruction window, has been added into the sequence $q_q^m(t)$.

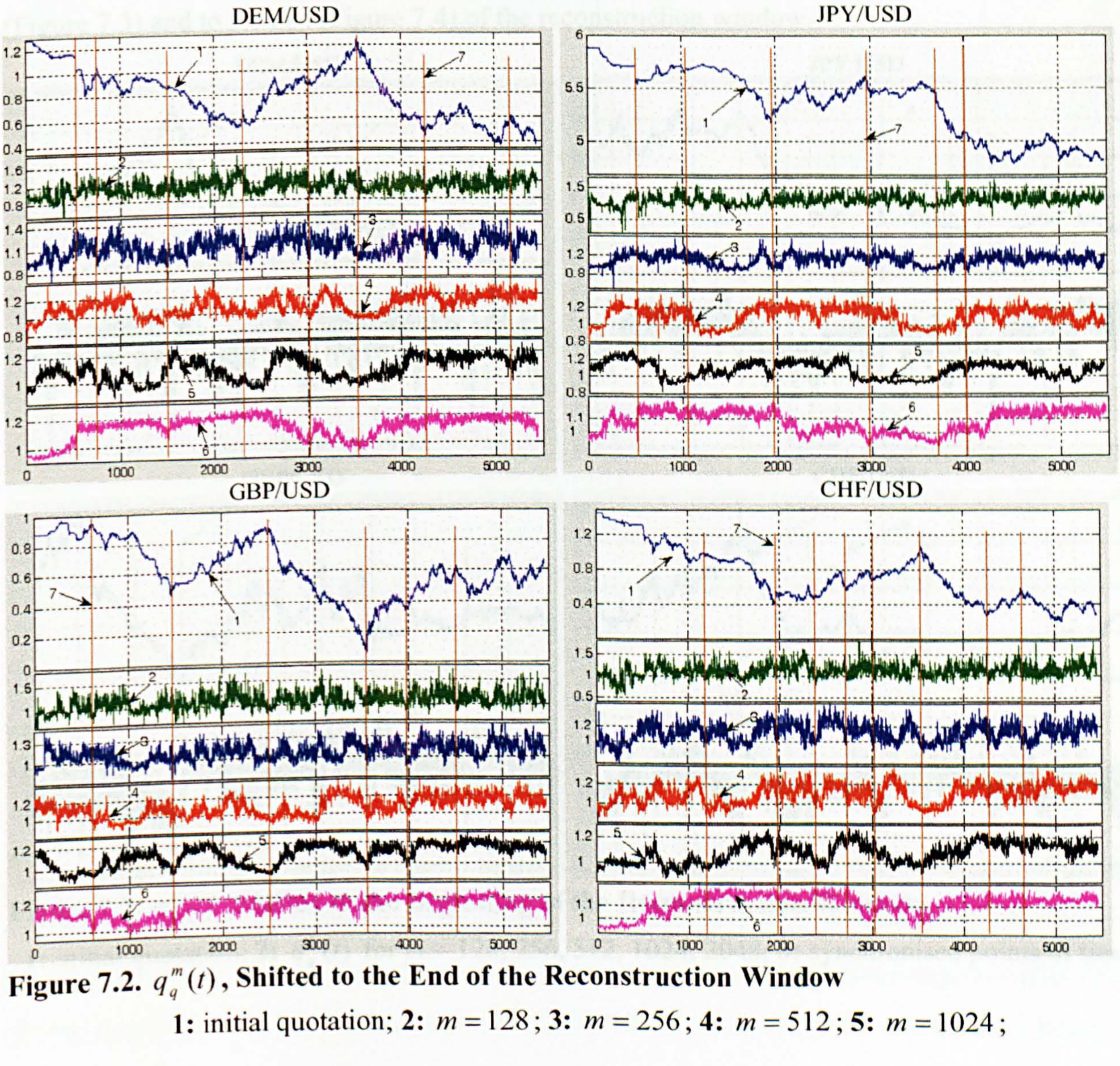


Figure 7.2. $q_q^m(t)$**, Shifted to the End of the Reconstruction Window**

1: initial quotation; **2:** $m = 128$; **3:** $m = 256$; **4:** $m = 512$; **5:** $m = 1024$;

6: $m = 2048$; **7:** synchronised points of time

Such behaviour of the parameters $q_q^m(t)$ allows for the results obtained to be considered in terms similar to the false nearest neighbours (FNN). So, if we are dealing with the real changes in the parameters $q_q^m(t)$ (true nearest neighbours), caused by significant changes in the system, then (with rare exceptions) these changes have to be observed in nearly all the reconstruction windows. At the same time, a false change in the parameters $q_q^m(t)$, associated with the exit of an anomalous value from the reconstruction window, is usually observed at different points of time. As a result, from the time series $q_q^m(t)$, for identification purposes, it is possible to propose the use of an averaged time series of the kind $q_q^{\Sigma}(t) = \dfrac{\left(q_q^{m_1}(t) + q_q^{m_2}(t) + \ldots + q_q^{m_k}(t)\right)}{k}$, where k is the number of time series $q_q^m(t)$, that significantly characterising the behaviour of the system. Figures 7.3 and 7.4 present the results of extracting $q_q^{\Sigma}(t)$, shifted correspondingly to the beginning (Figure 7.3) and to the end (Figure 7.4) of the reconstruction window.

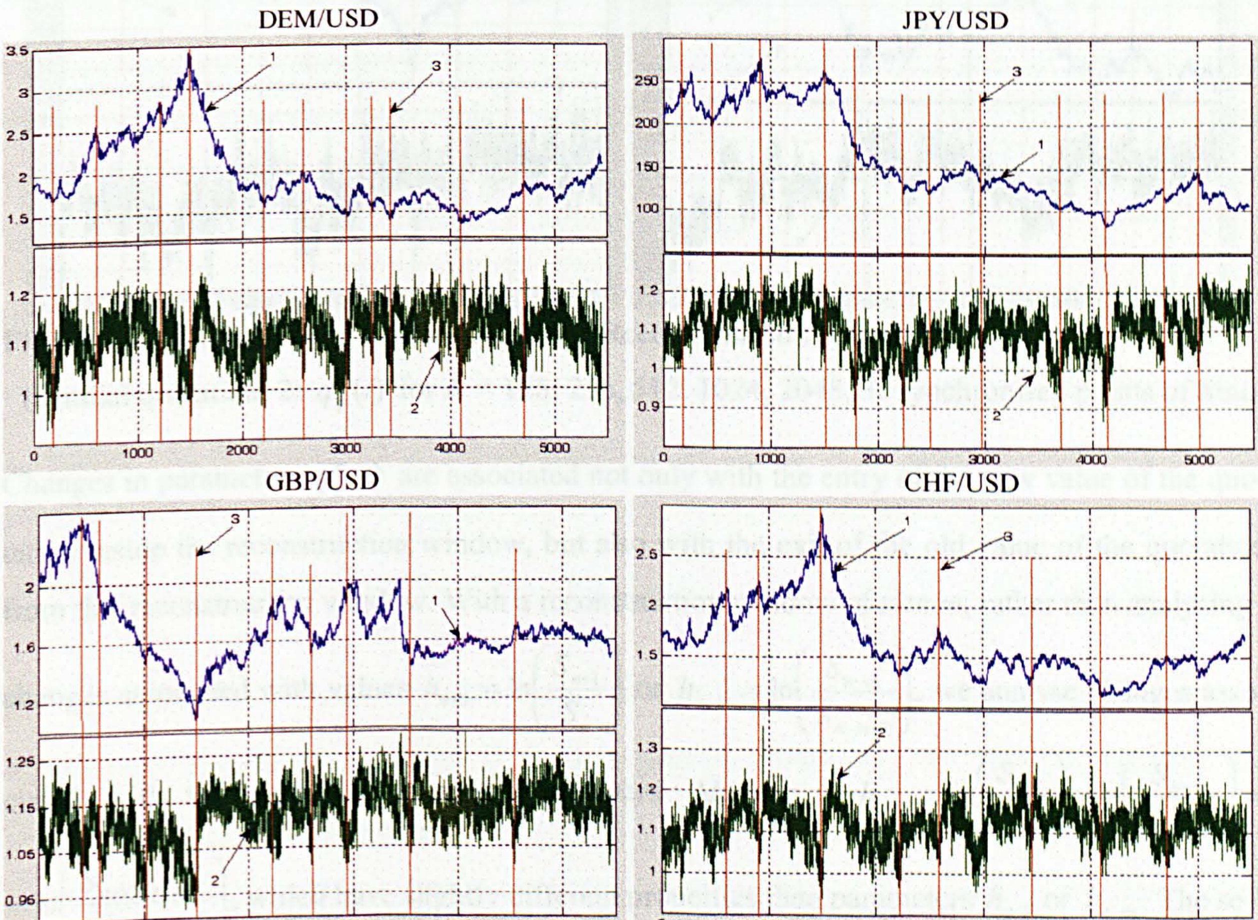


Figure 7.3. $q_q^{\Sigma}(t)$**, Shifted to the Beginning of the Reconstruction Window**

1: initial quotation; **2:** $q_q^{\Sigma}(t)$ for $m = 128, 256, 512, 1024, 2048$; **3:** synchronised points of time

Parameters $q_q^{\Sigma}(t)$ not only retain the identification properties of the parameters $q_q^{m}(t)$ for $q_q^{\Sigma}(t) \le 1$, but also exhibit similar properties both for $1 \le q_q^{\Sigma}(t) \le q_q^{m}$ and $q_q^{\Sigma}(t) \ge q_q^{m}$. False changes in parameters $q_q^{\Sigma}(t)$ are not usually observed.

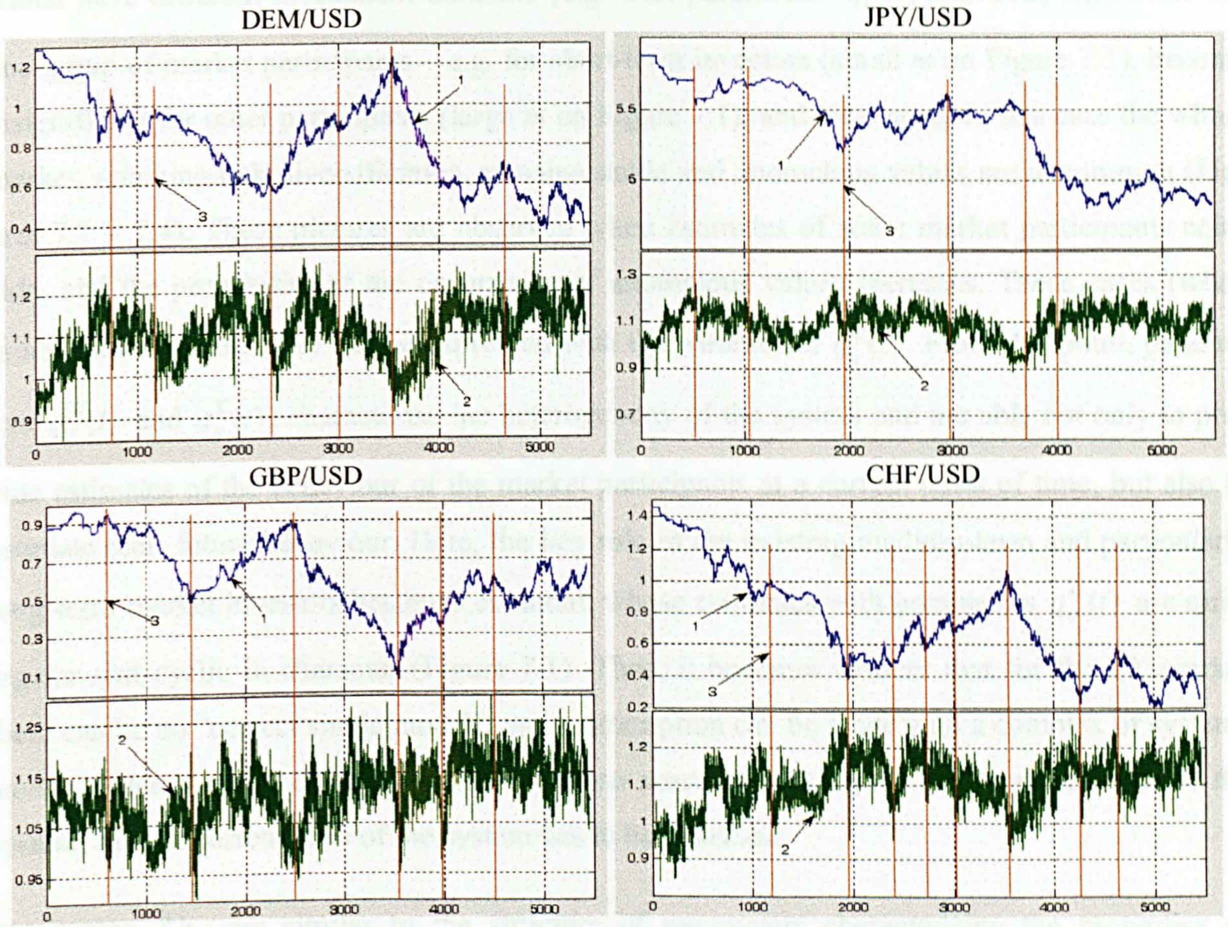


Figure 7.4. $q_q^{\Sigma}(t)$**, Shifted to the End of the Reconstruction Window**

1: initial quotation; **2:** $q_q^{\Sigma}(t)$ for m = 128, 256, 512, 1024, 2048; **3:** synchronised points of time

Changes in parameters $q_q^{m}(t)$ are associated not only with the entry of the new value of the quotation inside the reconstruction window, but also with the exit of the old value of the quotation from this reconstruction window. With a reconstruction window of size m, rather than analysing changes associated with values $h_{n+1} = \ln\left(\frac{S_{n+1}}{S_n}\right)$ or $h_{n-m} = \ln\left(\frac{S_{n-m}}{S_{n-m-1}}\right)$, we analyse changes associated with the combined change parameters $\Delta h_m = h_{n+1} - h_{n-m} = \ln\left(\frac{S_{n+1}}{S_n}\right) - \ln\left(\frac{S_{n-m}}{S_{n-m-1}}\right) = \ln\left(\frac{S_{n+1}S_{n-m-1}}{S_n S_{n-m}}\right)$, which have slightly different properties than parameters h_{n+1} or h_{n-m}. The sequence Δh_m has a finite memory, defined by m, which is constant for the reconstruction window, and accurately corresponds to the notion of a financial horizon. Then, using various values of m, for the current point of time it is possible to generalise the processes occurring on the FX-market from the different points of view of different investors. It is possible to classify the processes for both the whole FX-market (e.g. with parameters $q_q^{\Sigma}(t)$), and for various groups of participants, which have different investment horizons (e.g. with parameters $q_q^m(t)$). Events, significant for one group of market participants – e.g. for short-term investors (small *m* on Figure 7.1), become insignificant for other participants (large *m* on Figure 7.1), and vice versa. In this case the whole market, retaining risk diversification, remains stable and anomalous values are uncommon (Figures 7.1 – 7.4). These pictures are observed when estimates of some market participants coincide, and the probability of the occurrence of anomalous values increases. These cases (when group estimates coincide) can be expressed with the parameters $q_q^{\Sigma}(t)$. From this point, parameters $q_q^m(t)$ and $q_q^{\Sigma}(t)$, characterise the heterogeneity of the system and are able not only to provide estimates of the behaviour of the market participants at a current point of time, but also to estimate their future behaviour. Here, the key role of the existing medium-term and particularly long-term market investors becomes essential, whose estimates with parameters $q_q^m(t)$ are quite regular and cyclic in character (Figure 7.1). Thus, it becomes evident that for the FX-market there can be no "correct single model", and a description can be made with a complex of systems having different aims, among which there is no sense of looking for the "correct" state of the system, but the current state of the system has to be observed.

Parameters Δh_m are similar to the structure of parameters characterising the properties of changes of volatility for quotations. It has to be pointed out that the empirical volatility $\hat{\sigma}_n$ of the quotations can be considered as a statistical financial index, so the same methodology and approaches used for the analysis of quotations, can be applied to this index. Similarly to parameters h_n, it is (for instance) possible to introduce parameters $\hat{r}_n = \ln\left(\frac{\hat{\sigma}_n}{\hat{\sigma}_{n-1}}\right)$; $n \geq 2$. Following earlier findings and publications [6], parameters $\hat{r}_n$ with fractional structures change their values rapidly, which indicates that $\hat{\sigma}_n$ and $\hat{\sigma}_{n-1}$ are negatively correlated. This effect is perfectly observed and fully supported with the results of *R/S Analysis*, when $H \sim 0.2...0.4 < ½$. Taking into account all analogies made, it can be expected that parameters $q_q^m(t)$ also have fractional structure with $H < ½$. The results of *R/S Analysis* of $q_q^m(t)$ and $q_q^{\Sigma}(t)$ represented in Figure 7.5 and in Table 7.1 support this assumption. For the first approximation, parameters $q_q^m(t)$ and $q_q^{\Sigma}(t)$ can be considered as Gaussian, and therefore their negative correlatedness for the complete sequence, together with the observed self-similarity properties, provide further support to the fact that these sequences are fractional Gaussian noise with Hurst parameter $H < ½$.

$q_q^m(t)$, $m = 1024$ $\quad$ $q_q^{\Sigma}(t)$, $m = 128, 256, 512, 1024, 2048$

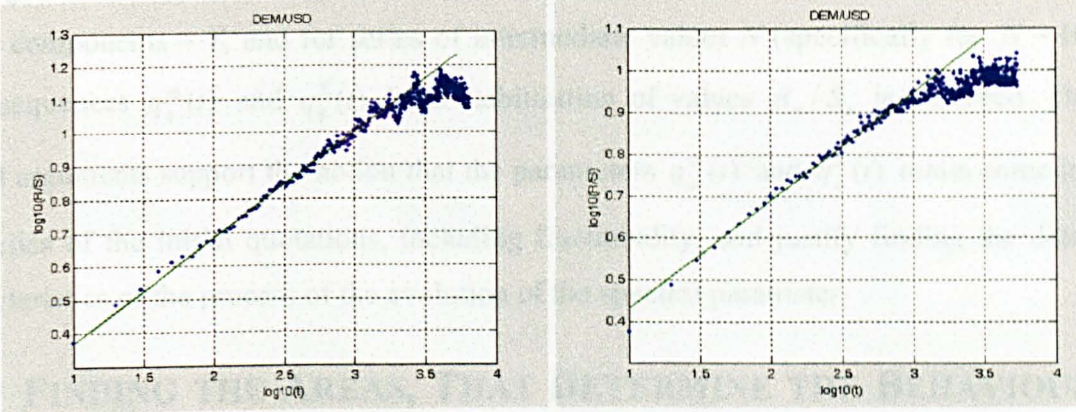


Figure 7.5. R/S Analysis of $q_q^m(t)$ and $q_q^{\Sigma}(t)$ for DEM/USD (for completed time series)

Table 7.1. R/S Analysis of $q_q^m(t)$ and $q_q^{\Sigma}(t)$ (for completed time series)

Data	$H_{\text{entry}}^{\Sigma}$	H_{exit}^{Σ}	H^{64}	H^{128}	H^{256}	H^{512}	H^{1024}	H^{2048}
DEM/USD	0.26	0.27	0.26	0.24	0.29	0.30	0.32	0.30
JPY/USD	0.27	0.27	0.28	0.28	0.31	0.30	0.33	0.30
GBP/USD	0.25	0.27	0.27	0.27	0.23	0.27	0.30	0.24
CHF/USD	0.27	0.27	0.27	0.27	0.25	0.27	0.29	0.24

In other words, it can be considered that sequences $q_q^m(t)$ and $q_q^{\Sigma}(t)$ retain some important properties of the parameters, characterising the behaviour of the initial quotations. It is worth mentioning that for fractional Brownian motion $\mathrm{E}(q_q^m(t))^2$ and $\mathrm{E}(q_q^{\Sigma}(t))^2$ increases as $|t|^{2H}$. Therefore, for relatively short time intervals ($N <\sim 30$), for *R/S Analysis* of $q_q^m(t)$ and $q_q^{\Sigma}(t)$, $H > ½$ (Figure 7.5), the spread of the parameters $|q_q^m(t)|$ and $|q_q^{\Sigma}(t)|$ is higher in comparison with the usual Brownian motion, and the proposed model of the evolution of the spectral parameter may not be well-defined, and it will be only possible to conclude that the processes have a fractional structure. For $N > 30$ (where $H < ½$) the spread of parameters $|q_q^m(t)|$ and $|q_q^{\Sigma}(t)|$ is smaller than for the usual Brownian motion. In the space $q_q^m(t)$ and $q_q^{\Sigma}(t)$ we are faced with a diffusion process, occurring in some fractional environment, where diffusion is slower than in the usual Brownian environment. Then, in terms of the method of limit estimates, the proposed model of the evolution of the spectral parameter can be considered as a working model, which characterises the intensive changes. When generating $q_q^{\Sigma}(t)$, the process of averaging, which includes the summing of parameters $q_q^m(t)$ for different m, this does not lead to $H \to ½$, which indicates the presence of correlation properties in the sequences. Sequences $q_q^m(t)$ and $q_q^\Sigma(t)$, as well as the initial quotations, have a number of properties, that could be perfectly observed with *R/S Analysis* (Figure 6.11). Among these, for instance, are the properties associated with the presence of cyclic components – *Y*, and for series of intermediate values N (specifically for $N \sim 1000$) for most sequences $q_q^m(t)$ and $q_q^\Sigma(t)$ local stabilisation of values R_n / S_n is observed. The represented arguments support the notion that the parameters $q_q^m(t)$ and $q_q^\Sigma(t)$ retain some important properties of the initial quotations, including fractionality, and justify finding the determining characteristics of the process of the evolution of the spectral parameter.

7.3 FINDING THE AREAS, THAT DETERMINE THE BEHAVIOUR AND DEVELOPMENT OF THE DYNAMIC SYSTEM

Finding characteristic parameters for the process of the evolution of the spectral parameter has to start by finding the main areas, that determine the behaviour and development of the considered dynamic system on the basis of analysing the relationships between the values of pairwise correlation coefficient $\rho^m(t)$ (of the direct and the inverse regressions) and the spectral parameter $q_q^m(t)$ itself. Let us establish the relationship $\rho^m(t) = \rho^m\left(q_q^m(t)\right)$ for the reconstruction window of different size m (Figure 7.6). For $m \leq 128$ the results show that for parameters $\rho^m(t)$ and $q_q^m(t)$ there is some stochastic relationship, which at first approximation can be considered as unique and almost linear for the whole data set. Although the considered extraneous fields have extensive data spread, but are relatively homogeneous, so this in turn allows us to analyse one unique area, using a uni-modal distribution. In other words, in terms of $q_q^m(t)$, the behaviour of the short-term investors is mostly homogeneous, and there are no drastic changes happening at different periods of time (Figures 6.27, 7.1 and 7.6).

As m increases a stochastic relationship tends to a functional relationship. The spread of data reduces, and it becomes more evident that in the extraneous fields some independent areas emerge. These areas determine the development of the dynamic system, each having its own characteristic properties (with the existing almost linear relationships inside this area). In particular, it is possible to choose at least three areas (Figure 7.6). The first area exists for $q_q^m(t) \sim 0.95$ and large $\rho^m(t) \sim 0.95$; and the second – for $q_q^m(t) > 1.06$ and small $\rho^m(t) \sim 0.8$. The third area links the first two areas and embraces nearly the whole range of changes of $\rho^m(t)$ and $q_q^m(t)$. If the first or the third areas can be considered as a single whole, then in some cases the determining parameters of the second area are quite distinctive from the same parameters obtained for the whole data set (Figures 6.27 and 7.6).

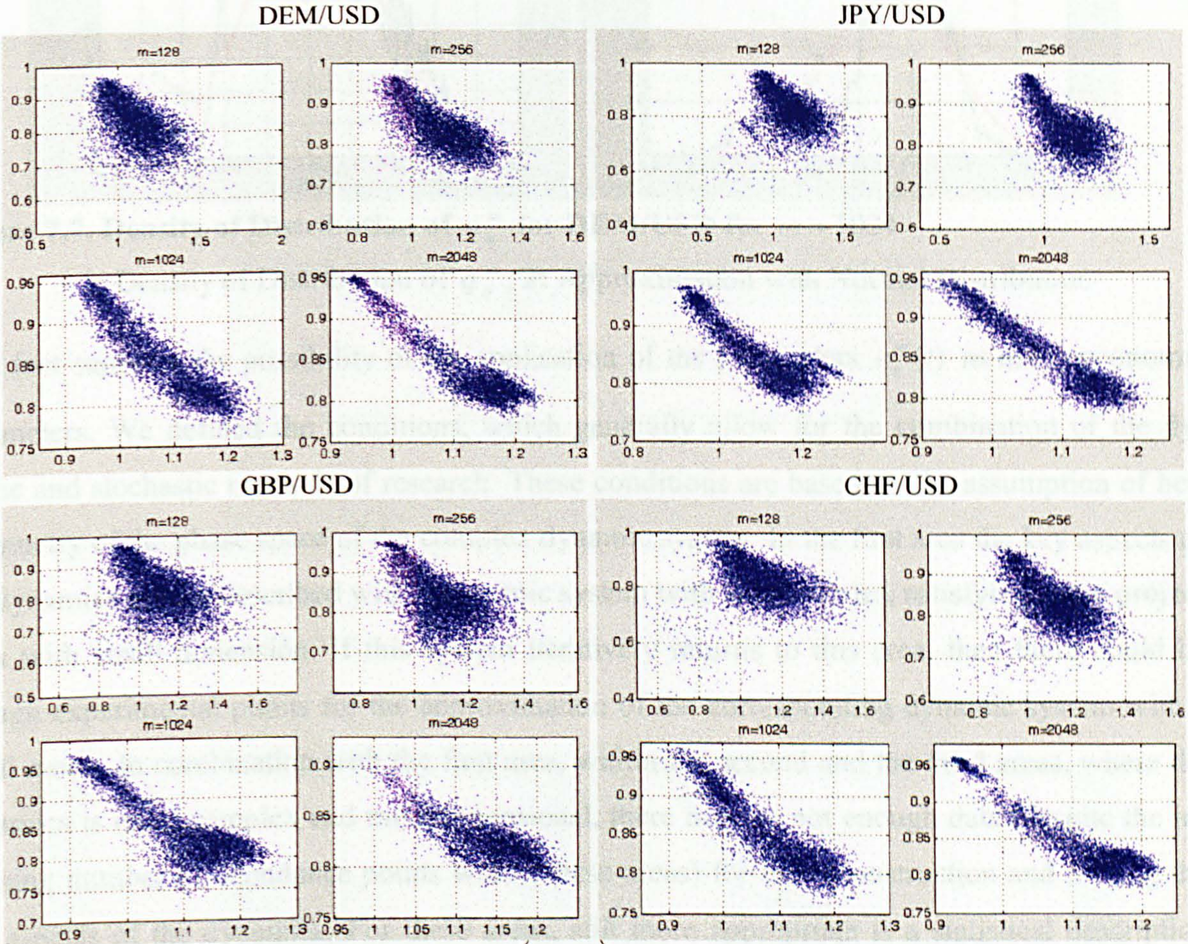


Figure 7.6. The Relationships $\rho^m(t) = \rho^m\left(q_q^m(t)\right)$ for Different Reconstruction Windows *m*

It becomes possible to argue for heterogeneity of the extraneous fields, and to consider each of these areas not with uni-modal distributions, but to analyse their interactions, applying, for instance, a poly-modal distributions. With increasing *m*, the first area remains more predictable ($\rho^m(t) \to 1$), but becomes less attended; while the second area, with the remaining high spread of data ($\rho^m(t) < 0.85$), becomes more and more attended. In other words, in terms of $q_q^m(t)$ the behaviour of the medium-term and particularly the long-term ($m \geq 1024$) investors change substantially (Figure 6.27). For instance, according to Figures 7.1 and 7.3, for DEM/USD ($N > 3000$) all drastic changes in investors' behaviour are typical of the mid-1990s (Figure 6.1, $N > 5000$), when such global processes were taking place, such as the introduction of the Euro currency, the unification of Germany, the collapse of the Soviet Union, etc. (Figure 7.7).

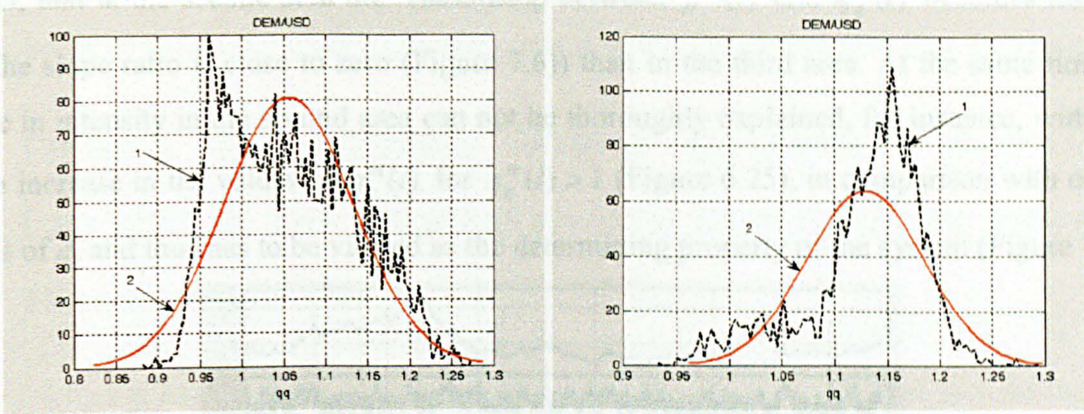


Figure 7.7. Density of Distribution of q_q^m for DEM/USD for $m = 1024$

1: Density of Distribution of q_q^m; **2:** Approximation with Normal Distribution

This fact supports the possibility of the application of the parameters $q_q^m(t)$ as macroeconomic parameters. We defined the conditions, which generally allow for the combination of the dynamic and stochastic methods of research. These conditions are based on the assumption of heterogeneity of the phase space of the complex dynamic system. In the first area the key aspects of the dynamics can be described with a dynamic system with small modes, satisfying some projections with small dimension. If this system iteratively returns to this area, then there could be enough experimental points for the approximation of the corresponding dynamic system with a small mode. In combination with the first area, within the second and the third areas, where the dynamics is more complex and multidimensional, there may be not enough data (despite the increasing number of attendance points within these areas) for the reconstruction and finding the key aspects of the dynamics. For these areas, still more appropriate is a statistical description with the use of uni-modal and poly-modal distributions. Practically, there could be any evolution in these areas, and thus the future of the dynamic system is unpredictable. According to this, parameters $\rho^m(t)$ can be viewed not only as a way to identify areas of the phase space of the system, but also as the parameters, characterising the predictability of new behaviour and the accuracy of possible forecasting (Figure 7.8).

Taking into account the statistical relationship between $\rho^m(t)$ and $q_q^m(t)$, it more and more trends to functional relationship in the first area with declining $q_q^m(t)$, it becomes evident why parameters $q_q^m(t)$, for $q_q^m(t) \leq 1$, could be considered as reliable identifiers for anomalous behaviour of quotations, resulting from changes in existing market tendencies. At the same time it becomes clear why for $q_q^m(t) > 1$ the degree of predictability of the systems remains low despite the fact, that in the second area the relationship between $\rho^m(t)$ and $q_q^m(t)$ becomes less intensive (the slope ratio is close to zero (Figure 7.6)) than in the third area. At the same time, such change in intensity in the second area can not be thoroughly explained, for instance, with only a certain increase in the values of $q_q^m(t)$ for $q_q^m(t) > 1$ (Figure 6.25), in comparison with other estimates of q, and thus has to be viewed as the determining property of the system (Figure 7.9).

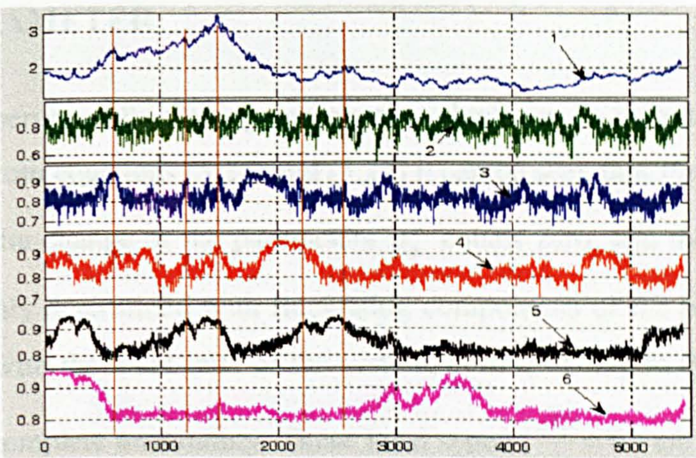


Figure 7.8. The Results of Finding $\rho^m(t)$ for DEM/USD

1: initial quotation; **2:** $m = 128$; **3:** $m = 256$; **4:** $m = 512$; **5:** $m = 1024$;
6: $m = 2048$; **7:** synchronised points of time

$\rho^m(t) = \rho^m\left(q_m^m(t)\right)$ $\rho^m(t) = \rho^m\left(q_\rho^m(t)\right)$

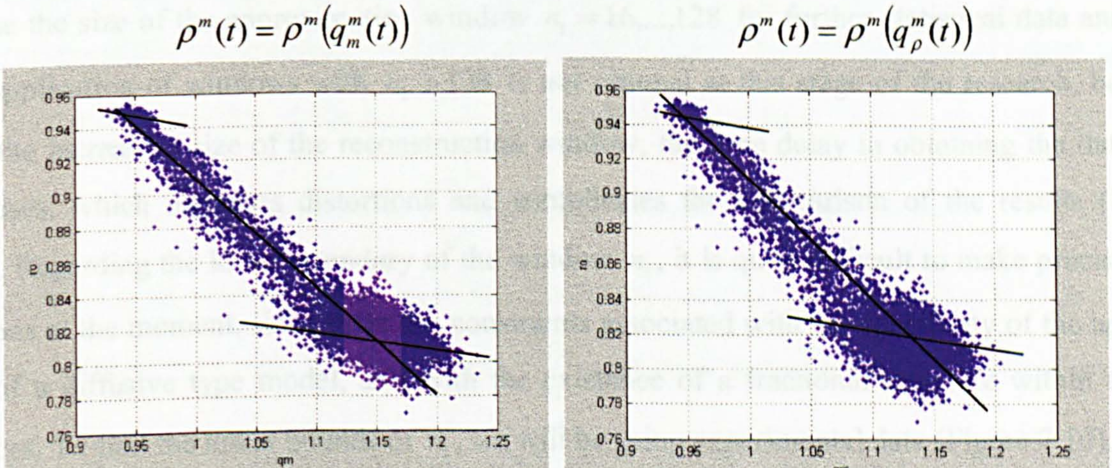


Figure 7.9. The Relationships $\rho^m(t) = \rho^m\left(q_m^m(t)\right)$, $\rho^m(t) = \rho^m\left(q_\rho^m(t)\right)$ for DEM/USD, $m = 2048$

Together with the possibility of an unambiguous classification of the main areas, determining the behaviour and the areas of development of the system, we have the possibility of experimentally finding the normalisation coefficients in the corresponding PDFs for these areas. This is important, because in the description of the process of the evolution of the spectral parameter for relatively large time intervals and/or the complete sequence, it is possible to turn from the use of more complex multi-modal distributions to the use of sets of relatively simpler single-mode distributions with corresponding weights (normalisation coefficients). However, it becomes evident that for classification and identification purposes we should not only consider the parameters $\rho^m(t)$, but have to look for other parameters, which determine the behaviour of the system.

7.4 Finding Parameters, Which Determine Systematic Changes in the Process of the Evolution of the Spectral Parameter

Firstly we turn to experimentally finding parameters, which can be included into the theoretical model (Chapter 5). From equations (5.24) and (5.27) it can be seen that for $A(q) = 0$ the average speed of the systematic change of the parameters q_q^m equals zero, and the evolution of the parameter q is thoroughly determined with fluctuating components of the process. The occurring changes are defined with the local state of the system, corresponding to the operation of a fractional market under complete uncertainty. These local states ($\frac{dq}{dt} = 0$) are quite easily identified with the parameters $q_q^m = q_q^m(t)$, after transition to finite differences $\frac{\Delta q_q^m}{\Delta t}$. For this, let us choose the size of the approximation window $n_1 = 16,...,128$ for further statistical data analysis. The application of windows with $n_1 > 128$ is not rational at this stage of the research, because with the increasing size of the reconstruction window, the time delay in obtaining the data also increases, which increases distortions and complicates the comparison of the results (Figure 7.10). Regarding the lower boundary of the window n_1, it is quite difficult to make precise conclusions at the moment, since there are constraints associated with the specificity of the application of a diffusive type model, and with the existence of a fractional structure within the sequences. To find the lower bounds of n_1, we will be using experimental data (Figure 7.10).

The results obtained indicate the existence of some patterns in the behaviour of parameters $\frac{\Delta q_q^m}{\Delta t}$. In particular, the proximity of the mean values to zero, the cluster effect for $n_1 = 0$, and the cyclic change of parameters from the means for other values of n_1 can be seen. It is possible to emphasise the finiteness of the changes and the possibility of fixing the changes in quotations' tendencies for $\frac{\Delta q_q^m}{\Delta t} = 0$, which for $\frac{dq}{dt} \approx 0$ are quite close to the sideways trend (Figure 7.10).

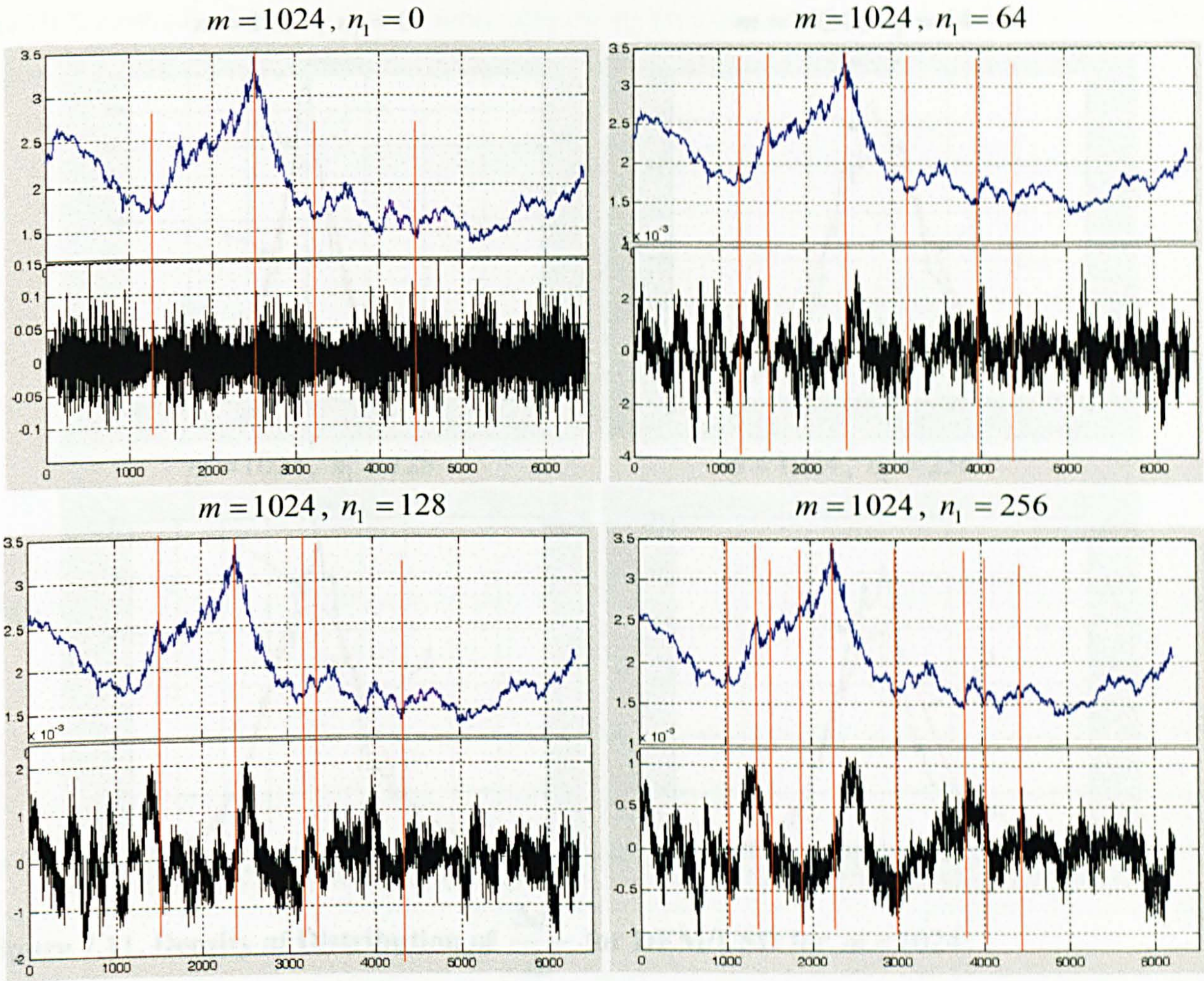


Figure 7.10. The Results of Finding $\frac{\Delta q_q^m}{\Delta t}$ for DEM/USD, $m = 1024$, $n_1 = 0, 64, 128, 256$

1: initial quotation; **2:** $\frac{\Delta q_q^m}{\Delta t}(t)$; **3:** synchronised points of time

Similar conclusions can be drawn from the results represented in Figure 7.11. It follows from Figures 7.10 and 7.11, that the parameters $\frac{\Delta q_q^m}{\Delta t}$ are quite homogeneous, and the mean value $\frac{\Delta q_q^m}{\Delta t} \approx 0$ only for the complete sequence, while generally for most periods of time $\frac{\Delta q_q^m}{\Delta t} \neq 0$ and has a well-defined sign. The PDF for these parameters are quite close (but not equal) to Normal distribution. For small n_1, these density distribution functions are more oblong, which indirectly indicates the fractional structure of these parameters, and the existence of after-effects (memory effects). When n_1 increases, distributions became more asymmetric, tending in shape to the distributions (5.39). All this allows us to conclude that usually changes in the parameters $q_q^m(t)$ are not entirely random, but have some patterns of behaviour, caused by changes occurring in the initial quotations.

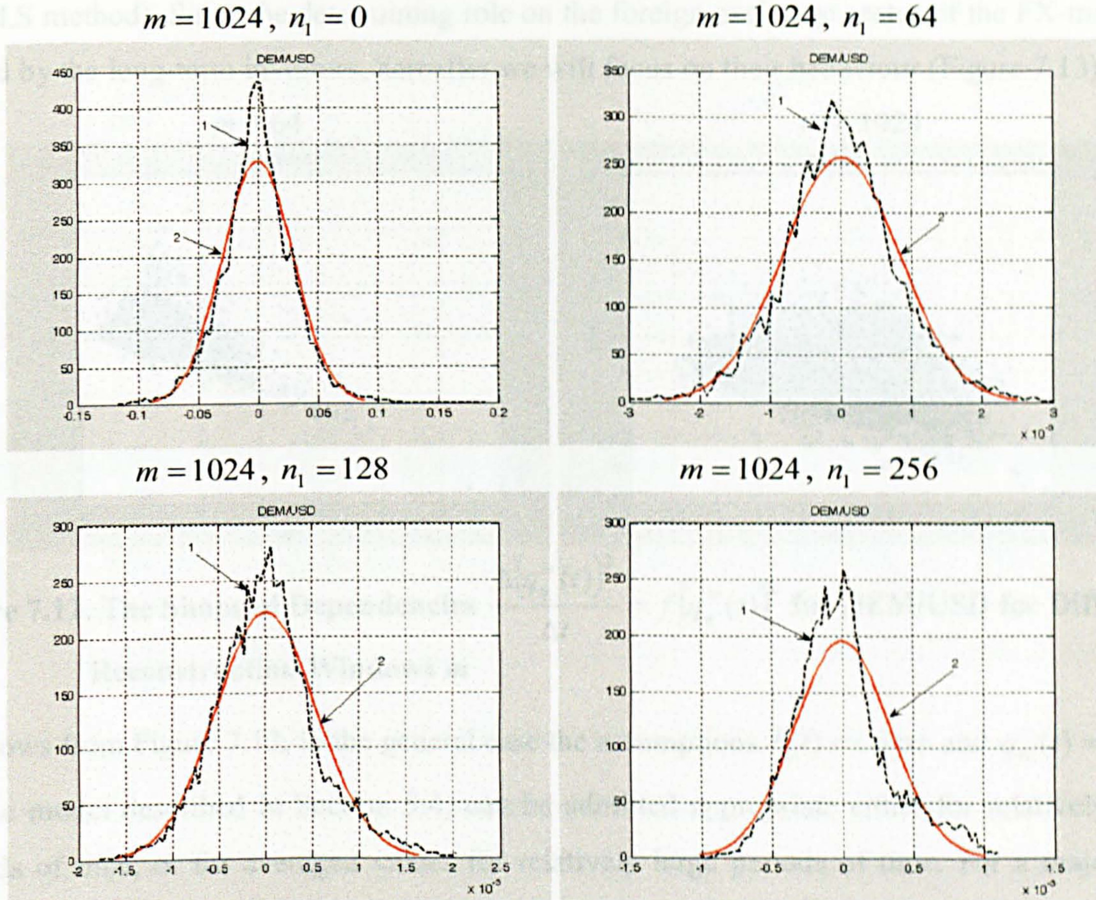


Figure 7.11. Density of Distribution of $\frac{\Delta q_q^m}{\Delta t}$ for DEM/USD for $m = 1024$

1: Density of Distribution of $\frac{\Delta q_q^m}{\Delta t}$; **2:** Approximation with Normal Distribution

Then for identification purposes for $n_1 = 16...128$, at the first stage of research, it is possible to consider equations, characterising average systematic changes of parameters $q_q^m(t)$ during the process of evolution. Let us turn to equation (5.31), and present it in the following form:

$$\frac{dq^2}{dt} = -2k(q^2 - q_{ec}^2) \tag{7.1}$$

Equation (7.1) is linear with respect to q^2 and $\frac{dq^2}{dt}$, which allows: $\frac{dx}{dt} = A_1 x + B_1$ to be written, where $A_1 = -2k$; $B_1 = 2kq_{ec}^2 = 2cq_0^2$ and $x = q^2$. So, if equation (7.1) holds, then in coordinates $\left(q^2; \frac{dq^2}{dt}\right)$ we have to obtain some extraneous fields with slope A_1 and drift B_1 (Figures 7.12).

In finite differences we get $\frac{\Delta x}{\Delta t} = A_1 x(t) + B_1$, which allows us to experimentally find the parameters $k(t) = -\frac{A_1(t)}{2}$ and $q_{ec}(t) = \sqrt{\frac{B_1(t)}{2k}}$ for different n_1 at various points of time (e.g. via the OLS method). Since the determining role on the foreign exchange sector of the FX-market is played by the long-term investors, hereafter we will focus on their behaviour (Figure 7.13).

$m = 64$ $\qquad$ $m = 1024$

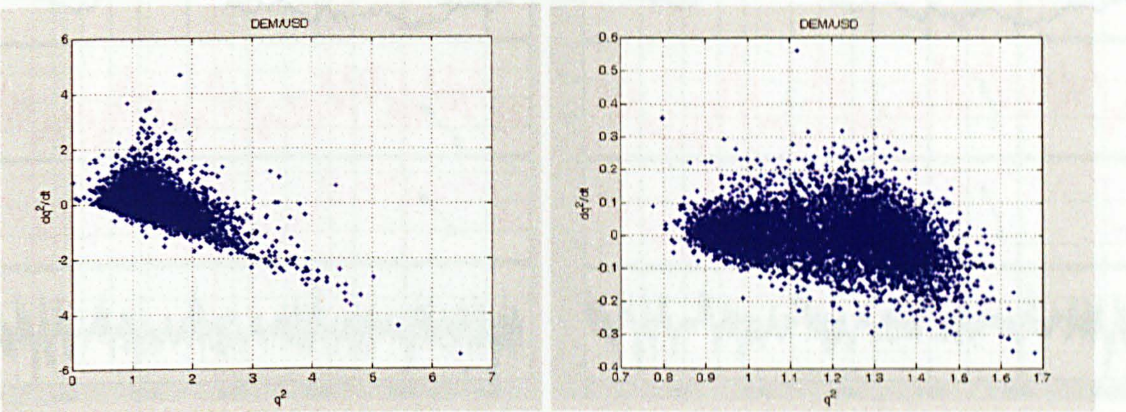
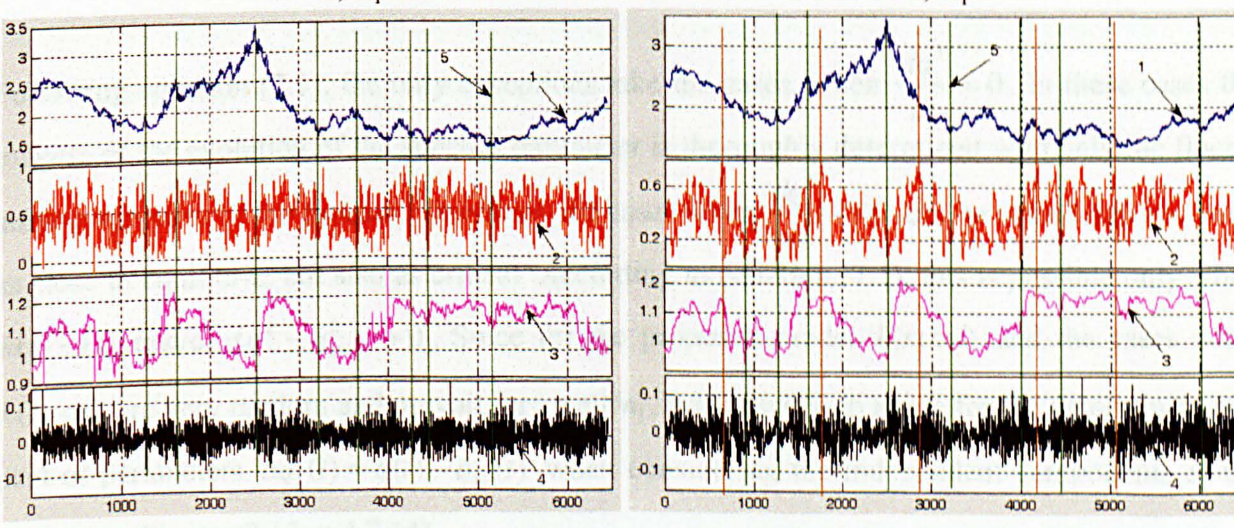


Figure 7.12. The Shape of Dependencies $\dfrac{\Delta\left(q_q^m(t)\right)^2}{\Delta t} = f\left(q_q^m(t)\right)^2$ **for DEM/USD for Different Reconstruction Windows *m***

It follows from Figure 7.13, in the general case the assumptions $k(t) = const$ and $q_{ec}(t) = const$, for the model described in Section 5.4, can be admitted appropriate either for relatively small periods of time, or for averaged values for relatively large periods of time. For a majority of cases $0 < k(t) < 1$, what corresponds to the assumptions of the model. Taking into account equation (5.36) and estimates obtained in Section 5.4, the last notion allows for the experimental confirmation of the validity of the research, based on fractional statistical analysis with the use of only daily quotations and without intraday data. So, it is already possible to state, that the values of the spectral parameter $q(t)$ "remember" the past even over relatively long time intervals (up to several days).

$m = 1024$, $n_1 = 16$ $\qquad$ $m = 1024$, $n_1 = 32$



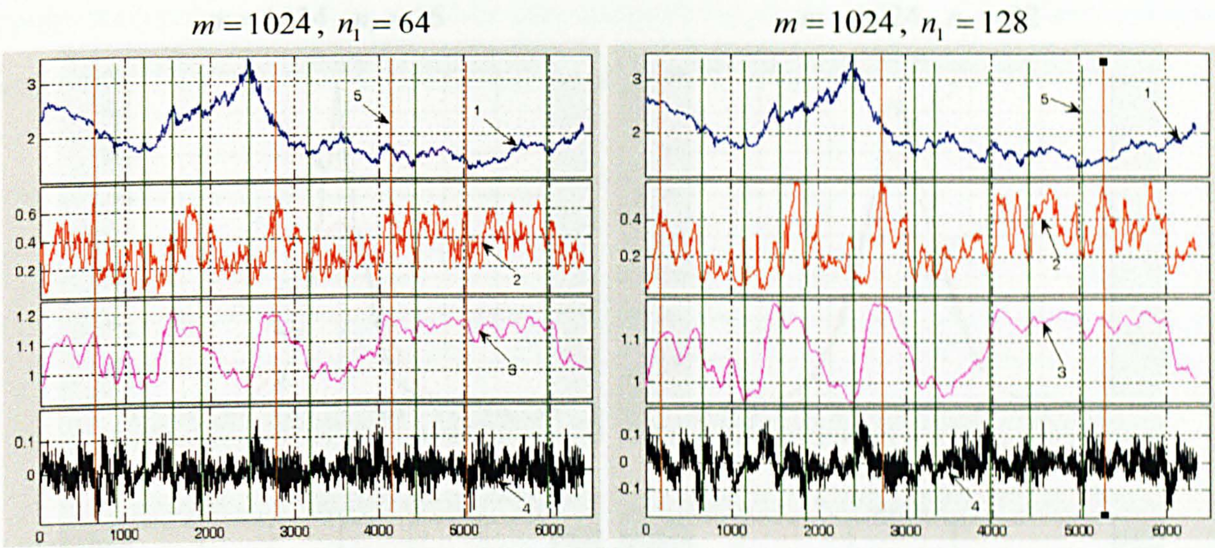


Figure 7.13. The Results of Finding $k(t)$, $q_{ec}(t)$, $\Delta q_1(t)$ for DEM/USD, $m = 1024$, $n_1 = 16$, 32, 64, 128

1: initial quotation; **2:** $k(t)$; **3:** $q_{ec}(t)$; **4:** $\Delta q_1(t)$; **5:** synchronised points of time

For relatively long time intervals, the averaged values of the parameters $q_{ec}(t)$, together with parameters $\rho^m(t)$, can be also used for identification purposes of the critical areas of the system. For instance, from Figure 7.13 it is possible to see that from the mid-1990s ($N \approx 4000$), the behaviour of the parameters $q_{ec}(t)$ has been changing dramatically. At the same time, it appears possible to provide a description of the critical areas. In particular, when the considering system for $N > 4000$, with frequently changing arbitrage attitudes of market participants (changes in $k(t)$), the impact of centralised governance is higher ($q_{ec}(t)$ are high) for more stable quotations and existing after-effects (usually $q_{ec}(t) > 1.1$). So it is possible to conclude that a systematic component of the process of the evolution of the spectral parameter exists (almost) permanently. Following equation (7.1), the only exceptions take the cases, when $\frac{dq^2}{dt} = 0$. In these cases the process of the evolution of the spectral parameter is thoroughly determined with only the fluctuating component of the process. Therefore, the cases when $\frac{dq^2}{dt} \to 0$ can be considered not only as close to equilibria, but also as critical. According to equation (7.1), this is possible only when $k(t) \to 0$ and/or $(q^2 - q_{ec}^2) \to 0$. Since for the proposed model $k(t) > 0$ and the cases when $k(t) < 0$ are very random and prevail for small n_1, it is possible to argue for the determining impact of parameters $\Delta q_1(t) = q(t) - q_{ec}(t)$ when considering the independent components of the processes (Figures 7.13 and 7.14).

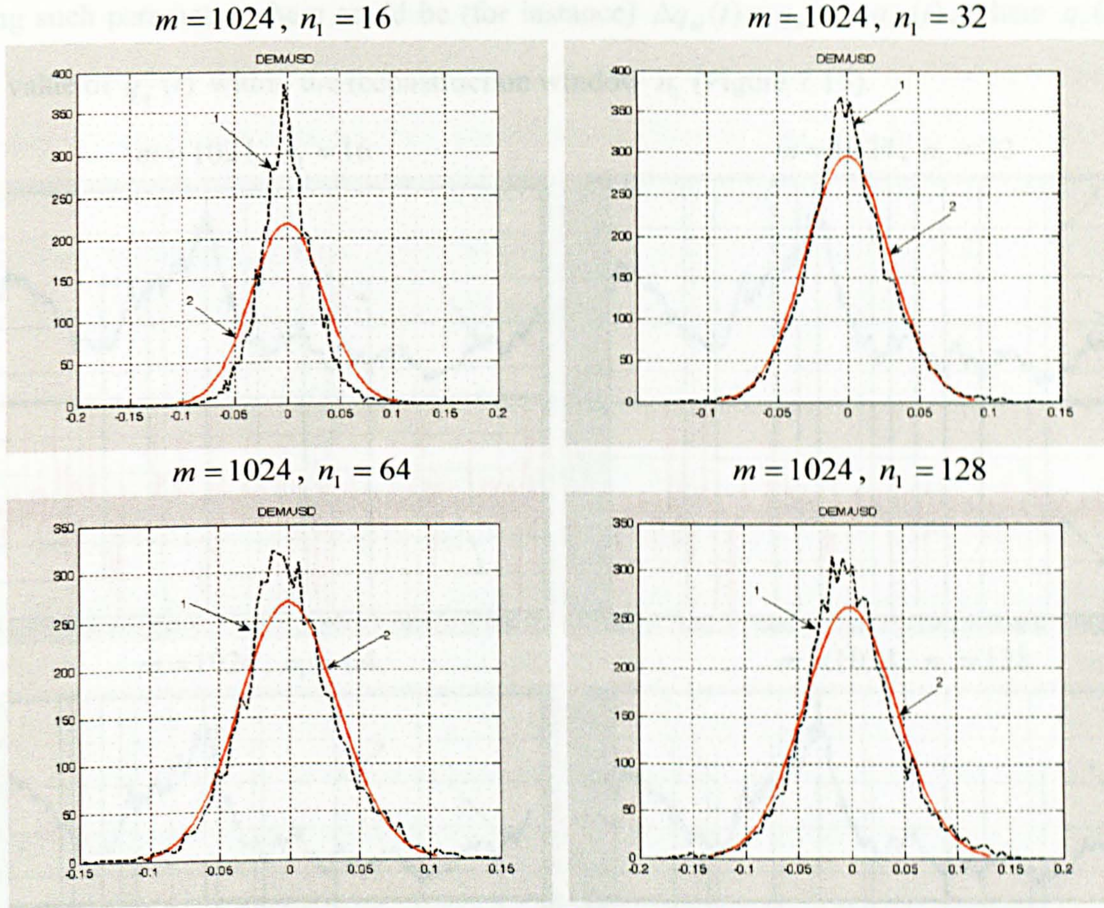


Figure 7.14. Density of Distribution of $\Delta q_1(t)$ for DEM/USD for $m = 1024$

1: Density of Distribution of $\Delta q_1(t)$; **2:** Approximation with Normal Distribution

Parameters $\Delta q_1 = \Delta q_1(t)$ are quite homogeneous, fluctuating (with some amplitude) around the equilibrium value $q_{ec}(t)$, which determines (according to the proposed model) the intensity of the impact of the control centre and/or of the external factors. For small n_1, the impact of fractional components of the process becomes more evident. When n_1 increases, this impact reduces, but the time delay in the obtained results increases. If the proposed model reflects reality, then for relatively large n_1 instead of a normal distribution of parameters Δq_1, we have to obtain a distribution with the type (5.39). The results represented in Figure 7.14 not only support this assumption, but also indicate that for $n_1 < 16$ the assumptions made for the proposed model would be mostly not satisfied. Thus, for $n_1 \approx 16,..,128$, the proposed model with the systematic component of the type (5.31) can be used as a fist approximation to describe the process of the evolution of the spectral parameter. It is significant that unlike the normal distribution where $\Delta q_1 = 0$, in the maximum of distributions $\Delta q_1 \neq 0$, regardless of the size of n_1 (Figure 7.14). This agrees with earlier conclusions (Sections 5.4 and 5.5), and for identification purposes allows the use of parameters, that determine the difference between the mode and mean values of distributions.

Among such parameters there could be (for instance) $\Delta q_m(t) = q_n(t) - q_{ec}(t)$, where $q_n(t)$ is a mean value of $q_q^m(t)$ within the reconstruction window n_1 (Figure 7.15).

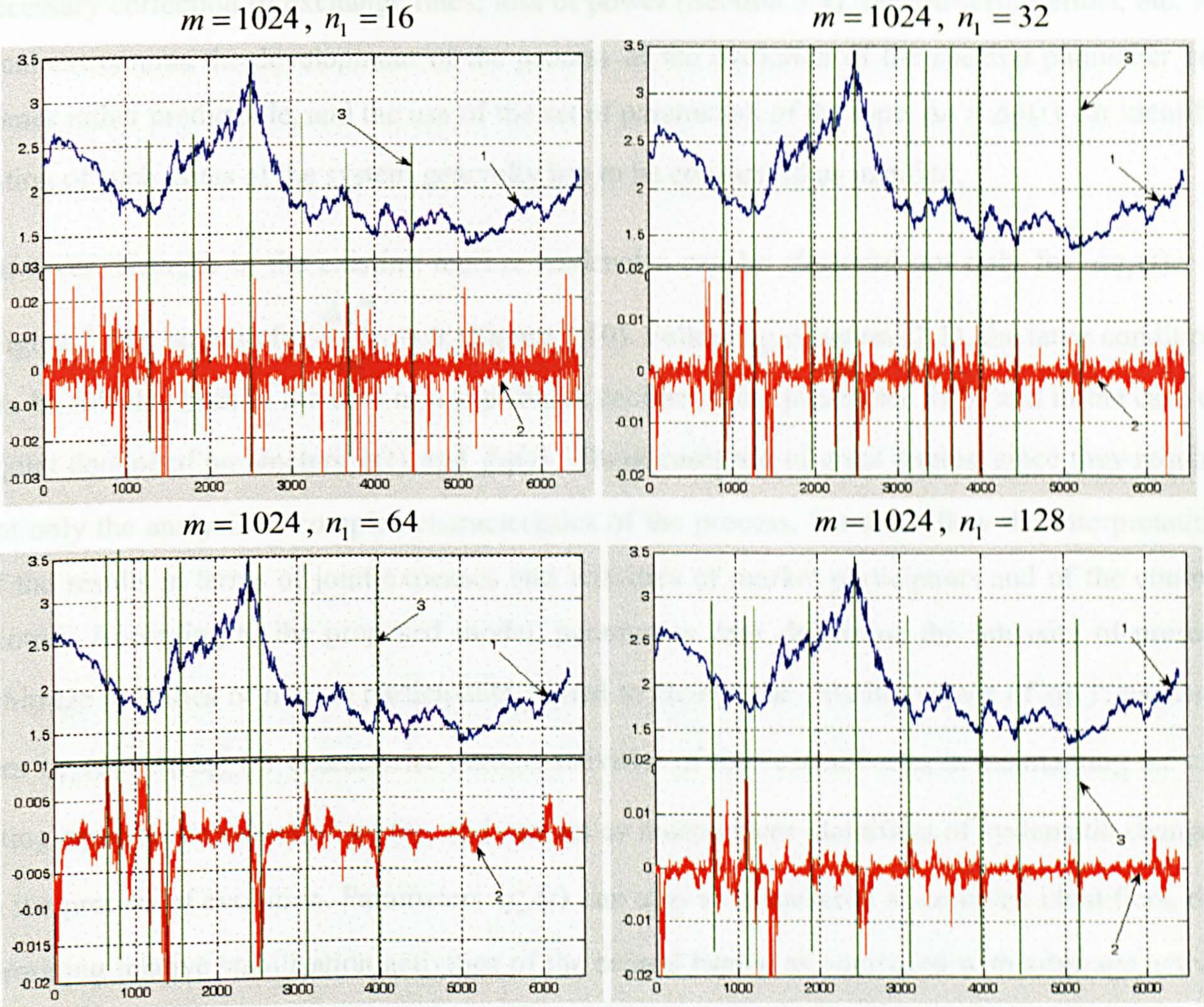


Figure 7.15. The Results of Finding $\Delta q_m(t)$ for DEM/USD, $m = 1024, n_1 = 16,32,64,128$

1: initial quotation; **2:** $\Delta q_m(t)$; **3:** synchronised points of time

Parameters $\Delta q_m(t)$ can be used not only for controlling the deviation scope of the obtained distributions from the normal (for $\Delta q_m(t) \approx 0$). They can also be applied for the classification of the stationary and non-stationary states of the system (for relatively high $\Delta q_m(t)$, and particularly for negative values of $\Delta q_m(t)$). The above results indicate, that to describe the changes in the systematic components of the process of the evolution of the spectral parameter, models of equilibrium dynamics are well suited. For identification and classification purposes it is also possible to use such non-dimensional parameters, as $\dfrac{\Delta q_1(t)}{q(t)}$, $\dfrac{\Delta q_1(t)}{q_{ec}(t)}$, $\dfrac{\Delta q_m(t)}{q_{ec}(t)}$, etc., that characterise deviation from the mean values, and represent the analogues to the variational components of the processes. It becomes evident and explainable why changes in the existing tendencies on the market can be observed when parameters of the type $\Delta q = \Delta q(t)$ reach the existing amplitudes of oscillation and/or $\Delta q = 0$. These explanations can be (for instance) provided in terms of the necessary correction of exchange rates, loss of power (Section 5.4), the clustering effect, etc. At local extremums the development of the process of the evolution of the spectral parameter becomes rather predictable, and the use of the set of parameters of the type $\Delta q = \Delta q(t)$ for identification of such states of the system generally has to be considered as justified.

However, changes in the existing market tendencies can be observed not only for $\Delta q_1(t) \approx 0$ (Figure 7.13), but also for $\frac{\Delta q_q^m}{\Delta t} \to 0$ (Figure 7.10). Following equation (7.1), the latter condition can be fulfilled both in the case of a significant decline of the parameter $k(t)$, and in the case of a joint decline of parameters $k(t)$ and $\Delta q(t)$. These cases are of great interest since they require not only the analysis of complex characteristics of the process, but also allow the interpretation of the results in terms of joint expenses and activities of market participants and of the control centres. According to the proposed model, parameters $k(t)$ determine the intensity of present arbitrage activities of market participants, aimed to change the existing values of $q(t)$; parameters $cq_0(t) = k(t)q_{ec}^2(t)$ characterise current activities of the central banks in maintaining the existing equilibrium values of $q_{ec}(t)$, and can act as independent identifiers of systematic changes in the process of evolution. Parameters $q_{ec}^2(t)$ can also be considered as complex identifiers, determining relative stabilisation activities of the central banks, as compared with arbitrage activities of market participants (Figure 7.16). Then, changes in market tendencies have to be expected not only for extremum values of $q_{ec}^2(t)$, but also when $q_{ec}^2(t) = 1$.

From the point of view of expenses and activities minimisation, central banks have to undertake major changes and make amendments to the parameters $q_{ec}(t)$ either when values of $k(t)$ are relatively small, or during the process of their decline. For relatively large values of $k(t)$, or when they increase, activities of the central banks have to be focused on stabilisation and maintenance of the existing equilibrium value of $q_{ec}(t)$. Similarly, for market participants, the main activities resulting in changes of $k(t)$ are better undertaken for relatively stable values of $q_{ec}(t)$, since when the parameters $q_{ec}(t)$ change substantially, all changes may turn out to be ineffective. Changes in the intensity of arbitrage activities of market participants are considered effective when responses of the central banks are insignificant, i.e. when value of $k(t)q_{ec}^2(t)$ decline sharply and turn out to be quite small for given period of time. Exactly after this, changes in the existing tendencies of the behaviour of the considered quotations should be expected. With increasing $k(t)q_{ec}^2(t)$, activities of the central banks and of market participants ($k(t)$ increases) should produce a stabilisation effect. Then, as $k(t)q_{ec}^2(t)$ increases, there have to be signs of stabilisation and/or the sideways trend in quotations. The emergence of anomalous behaviour in quotations has to be expected for any subsequent declines in $k(t)q_{ec}^2(t)$ (Figure 7.16), including sharp drops of $k(t)$ (Figure 7.11).

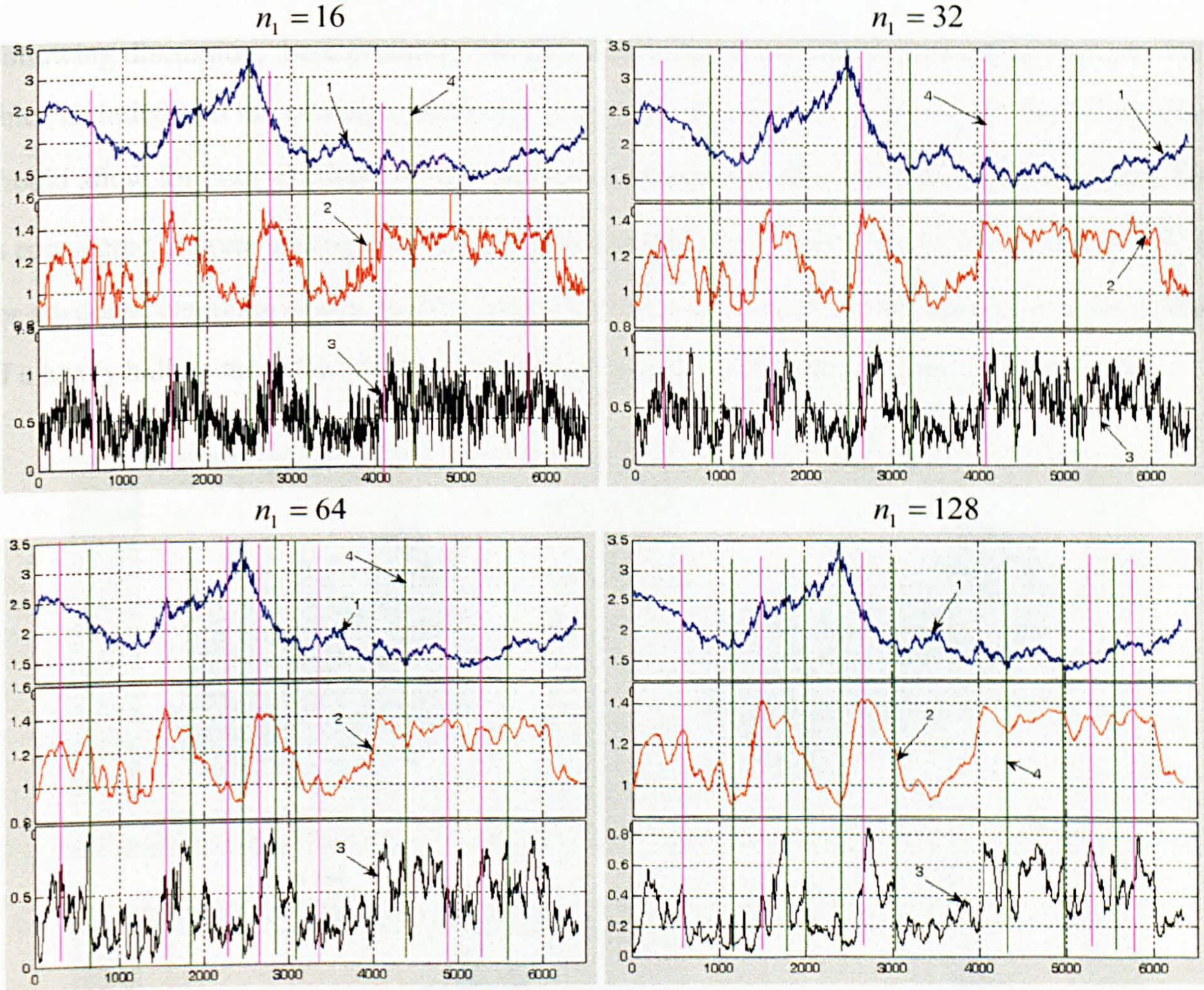


Figure 7.16. The Results of Finding $q_{ec}^2(t)$, $k(t)q_{ec}^2(t)$ for DEM/USD, $m = 1024$, $n_1 = 16, 32, 64, 128$

1: initial quotation; **2:** $q_{ec}^2(t)$; **3:** $k(t)q_{ec}^2(t)$; **4:** synchronised points of time

Thus, we have obtained another complex parameter, which not only corresponds to the proposed model, but also has an entire practical meaning: $k(t)q_{ec}^2(t) = cq_0^2(t)$ is the characteristic of intensity of the stabilisation activities of the central banks. Using this complex criterion, it is possible to explicitly interpret the results obtained. Moreover, since the behaviour of the control centres is basically determined by the external factors, which are hardly predictable in the long run, then activities of the central banks could be also considered as unpredictable in the long-term. As a result, the behaviour of the system (at least) in the long run allows for a consideration of the FX-market as the controlled system, but controlled only by the CBs, which entirely coincides with the main characteristics of the market. Thus, it is possible to conclude that with increasing time, the risk for market participants on FX-market is also increasing. Therefore, they have to prefer short-term trading, when during local points of time the impact of the control centre is relatively stable and rather predictable, at least from a probabilistic point of view.

Following discussion, there generally has to be a non-linear statistical relationship between variables included into the complex parameter $k(t)q_{ec}^2(t)$. With minor reservations, this relationship should allow for both physical and geometrical interpretation. For instance, if the parameter $k(t)$ is considered as some analogue to the potential energy of the system, then in coordinates $\left(q_{ec}^2;k\right)$ for adequate averaging degree (sufficiently high values of n_1), we get the analogue of the motion of a heavy ball on the sides of a horizontal pipe, which can change with time (Figure 7.17).

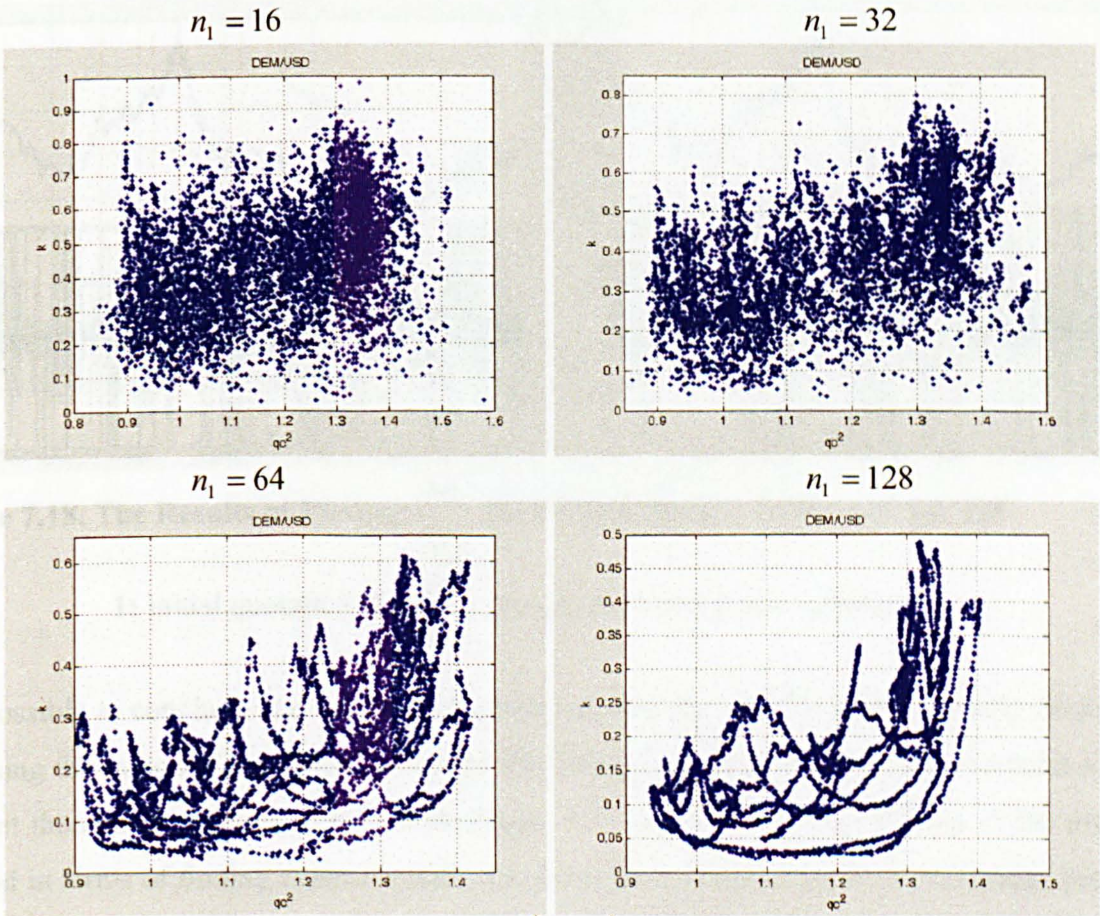


Figure 7.17. The Results of Finding $k(t) = f\left(q_{ec}^2(t)\right)$ for DEM/USD, $m = 1024$, $n_1 = 16$, 32, 64, 128

If to do some scaling, e.g. to introduce the scale $k = c_1 k$, and to hypothesise that ascending/descending of this ball on the side (i.e. rise/fall of k) of this pipe happens momentarily, we can get the classical brachistochrone problem, and then for some period of time it is possible to find the trajectory of this motion (the side view of the scaled pipe) in the form of a cycloid. Then, the trajectory of the whole motion can be represented as a set of cycloids, which could be affine and similar to each other. This hypothesis could be very interesting and effective. However, we are interested in it only from the point of view that such trajectories have to have return points, corresponding to local maxima of parameters $k(t)$, and, consequently, to some local maxima of the products $k(t)q_{ec}^2(t)$, which can be applied for identification purposes.

In the neighbourhood of the return points, parameters $\dfrac{dk}{dq_{ec}^2}$ have to change their signs. The position of the return point in time has to be slightly ahead of the corresponding crisis situation and the emergence of anomalous values, which already allows for the possibility of forecasting (Figure 7.18).

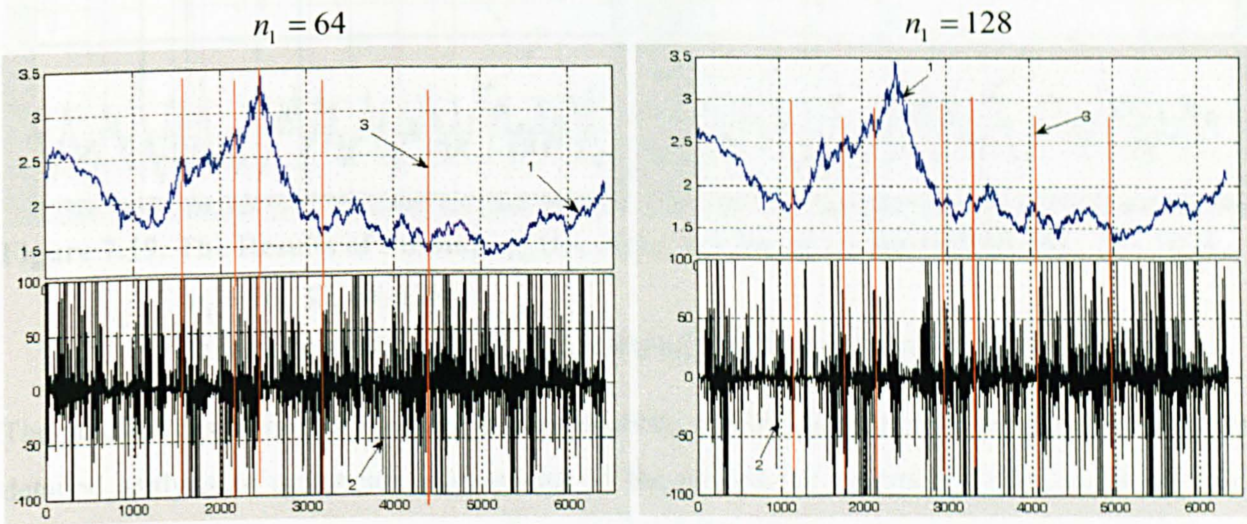


Figure 7.18. The Results of Finding $\dfrac{\Delta k}{\Delta q_{ec}^2}$ for DEM/USD, $m = 1024$, n_1 = 64, 128

1: initial quotation; **2:** $\dfrac{\Delta k}{\Delta q_{ec}^2}$; **3:** synchronised points of time

It is possible to conclude that we have obtained not only the tool for experimentally finding and analysing the systematic components of the occurring processes, but also define a basis for subsequent theoretical research in this field. Figure 7.19 describes the possibilities of the proposed method in terms of finding experimentally the systematic components of the occurring processes applied to the example of finding the complex characteristics of the process of the evolution of parameters $q_q^{\Sigma}(t)$.

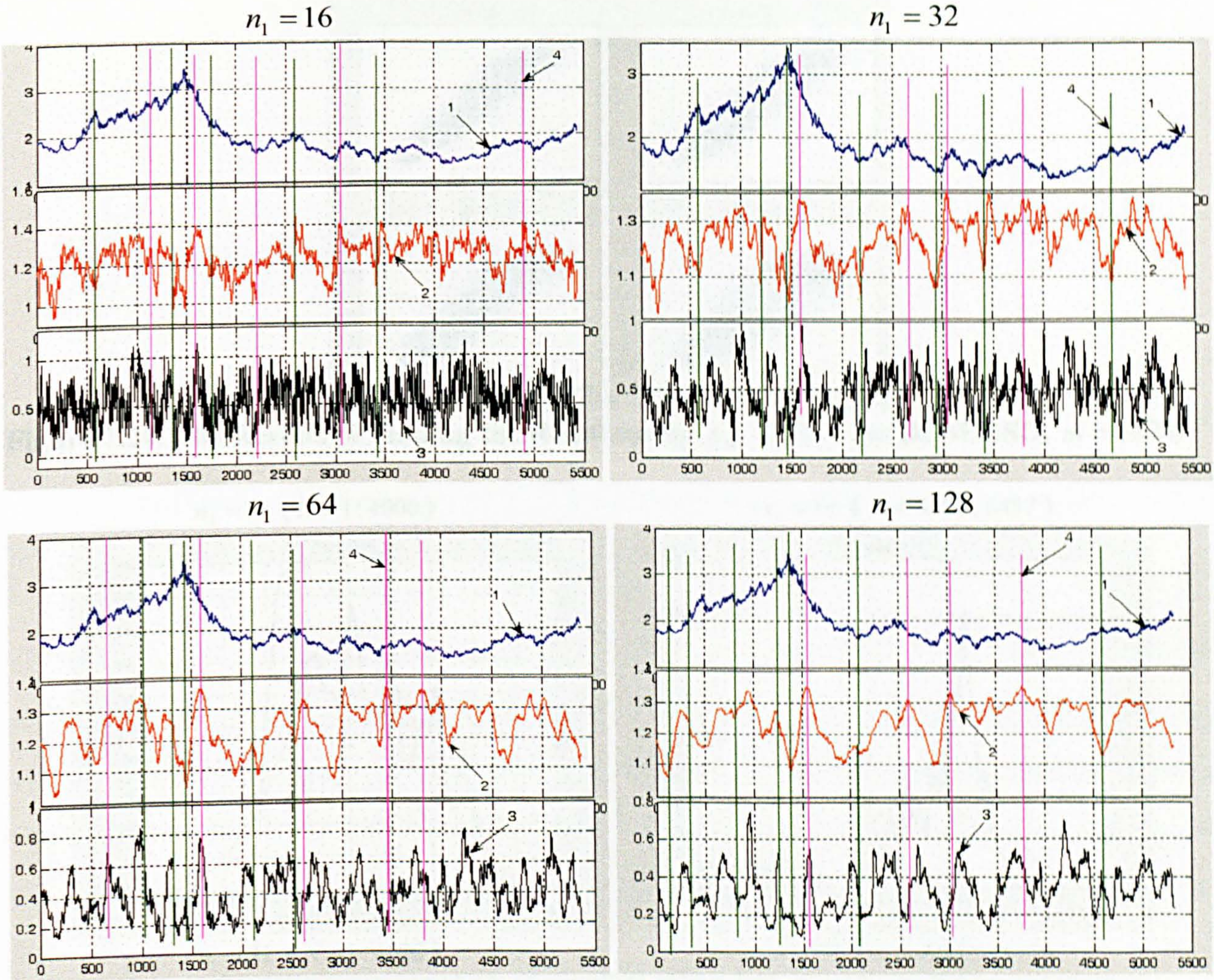


Figure 7.19. The Results of Finding $q^2_{ec}(t)$, $k(t)q^2_{ec}(t)$ for $q^{\Sigma}_q(t)$ for DEM/USD, $m = 1024$, $n_1 = 16, 32, 64, 128$

1: initial quotation; **2:** $q^2_{ec}(t)$; **3:** $k(t)q^2_{ec}(t)$; **4:** synchronised points of time

The possibilities of further theoretical investigations can be illustrated on the example of a more detailed analysis of systematic components of the process of evolution. From equation (7.1) it follows that, for $q_{ec} = const$, due to the presence of the fluctuation component, in coordinates $(q; q_{ec})$ we have to define the extraneous fields with some slope coefficient close to zero. If for a local time interval and adequate averaging degree, this is really possible, then for the complete sequence this does not hold (Figure 7.20). This case experimentally supports the earliest theoretical assumption that in the general case there could be quite a lot of local equilibria states of the FX-market (Figures 7.20 and 7.21), where N correspond to Figure 7.13.

Figure 7.20 also indicates that between parameters q_{ec} and q there has to be some stochastic relationship, which, to a first approximation, can be considered as unique and almost linear for the whole set of data.

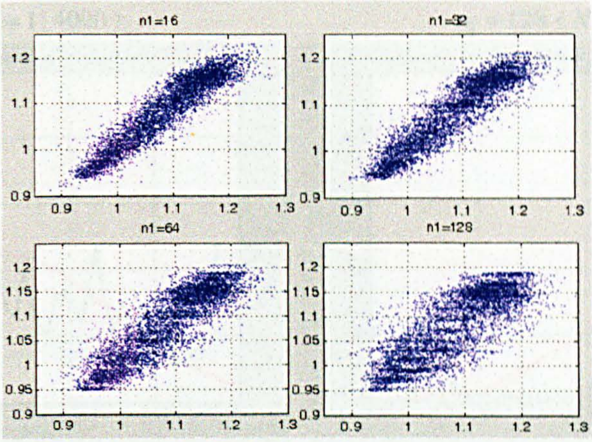
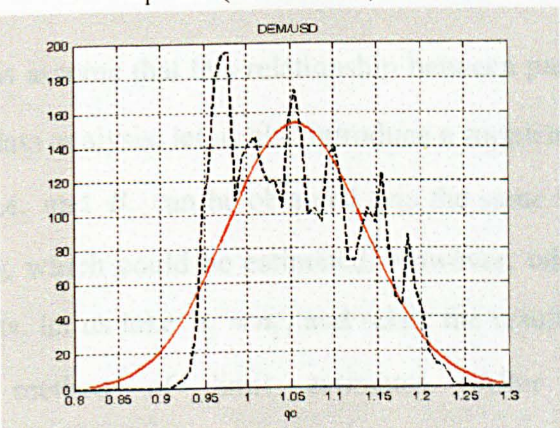
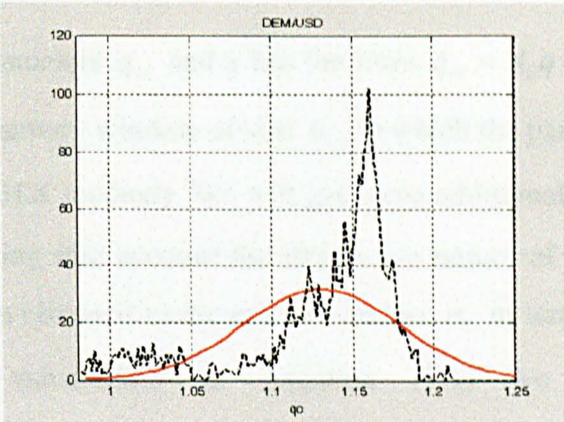


Figure 7.20. The Results of Finding the Relationship $q_{ec} = f(q)$ for DEM/USD, $m = 1024$

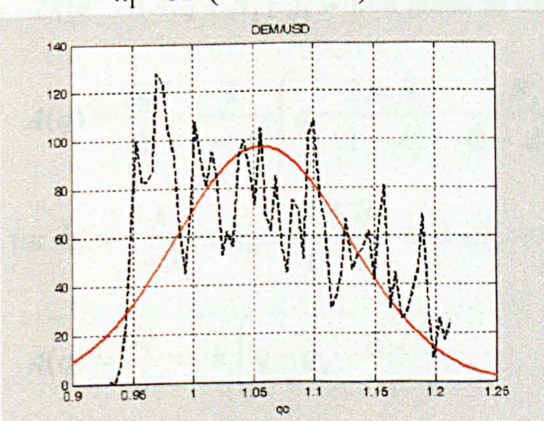
$n_1 = 16$ ($N = 1:4000$)



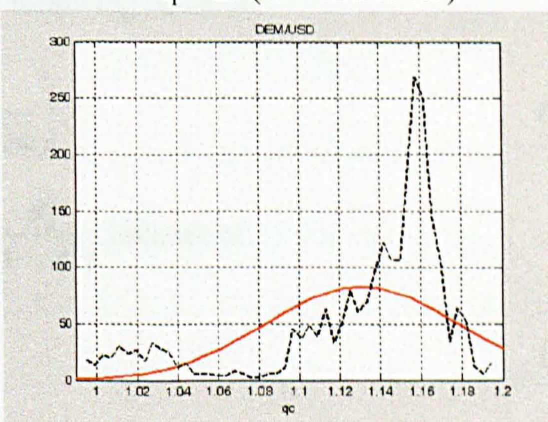
$n_1 = 16$ ($N = 4000:6457$)



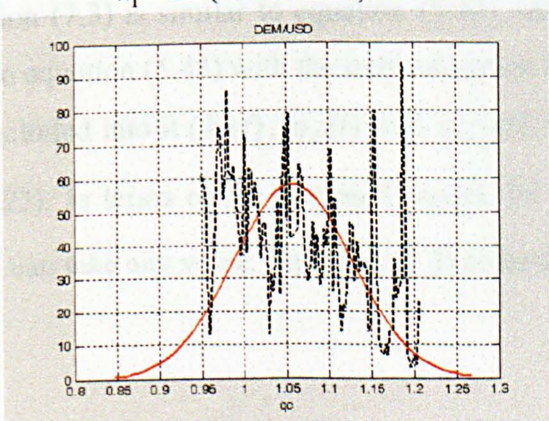
$n_1 = 32$ ($N = 1:4000$)



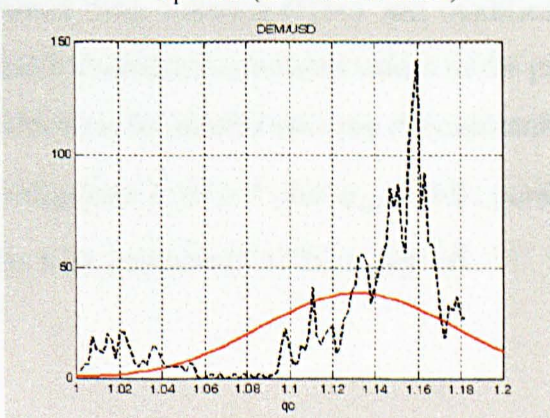
$n_1 = 32$ ($N = 4000:6441$)



$n_1 = 64$ ($N = 1:4000$)



$n_1 = 64$ ($N = 4000:6409$)



$n_1 = 128$ ($N = 1:4000$) $n_1 = 128$ ($N = 4000:6345$)

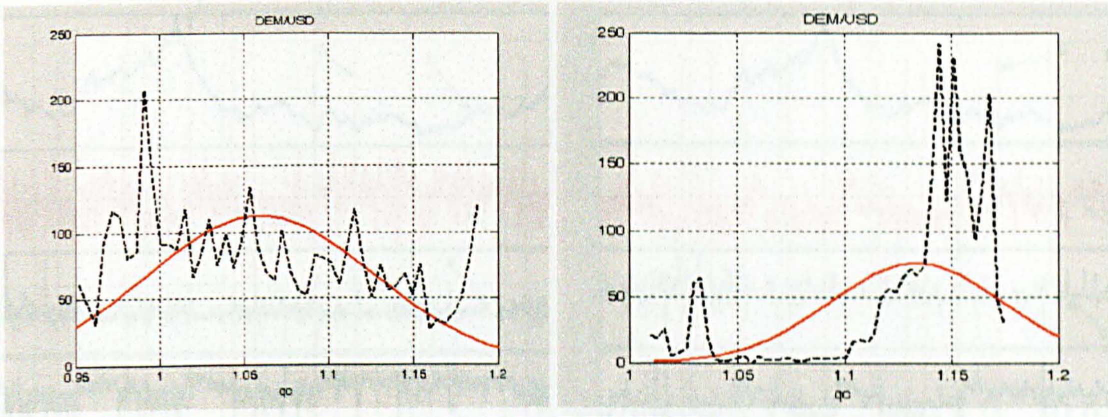


Figure 7.21. Density of Distribution of q_{ec}^m for DEM/USD for $m = 1024$

1: Density of Distribution of q_{ec}^m; **2:** Approximation with Normal Distribution

Let us assume that this relationship between parameters q_{ec} and q has the form $q_{ec} = A_2 q + B_2$. For data analysis, let us also introduce a supplementary window of size n_2, in which the parameters A_2 and B_2 can be obtained (via the same OLS method). We will get some additional time delay, which could be estimated. However, taking into account the illustrative nature of these results, let us take $n_2 = n_1$, and relate the results obtained to the end of window n_2 in terms of the method of limit estimates. After substitution in equation (7.1) we get: $\frac{dq^2}{dt} = -2k\left(q^2 - (A_2 q + B_2)^2\right)$, which leads to the following equation:

$$A(q) = \frac{dq}{dt} = \frac{k}{1 - A_2^2}\left(q - \frac{2A_2B_2}{1 - A_2^2} - \frac{B_2^2}{(1 - A_2^2)q}\right), \tag{7.2}$$

which for $k_i = \frac{k}{1 - A_2^2}$; $q_{ie} = \frac{2A_2B_2}{1 - A_2^2}$ and $q_{iec}^2 = \frac{B_2^2}{(1 - A_2^2)}$ reduces to:

$$A(q) = \frac{dq}{dt} = -k_i\left(q - q_{ie} - \frac{q_{iec}^2}{q}\right). \tag{7.3}$$

Equation (7.3) is similar to equation (5.48), obtained from equation (5.31), and modified similarly to equation (5.44) with the only exception that it characterises present values of the parameters included into it ($k_i(t)$, $q_{ie}(t)$ and $q_{iec}(t)$), which in the general case are not constants (Figure 7.22). In terms of the proposed model, for obligatory $k_i(t) \geq 0$ and $q_{iec}(t) \geq 0$, parameters $q_{ie}(t)$ can take any value. Equation (7.3) coincides with equation (7.1) for $q_{ie}(t) = 0$.

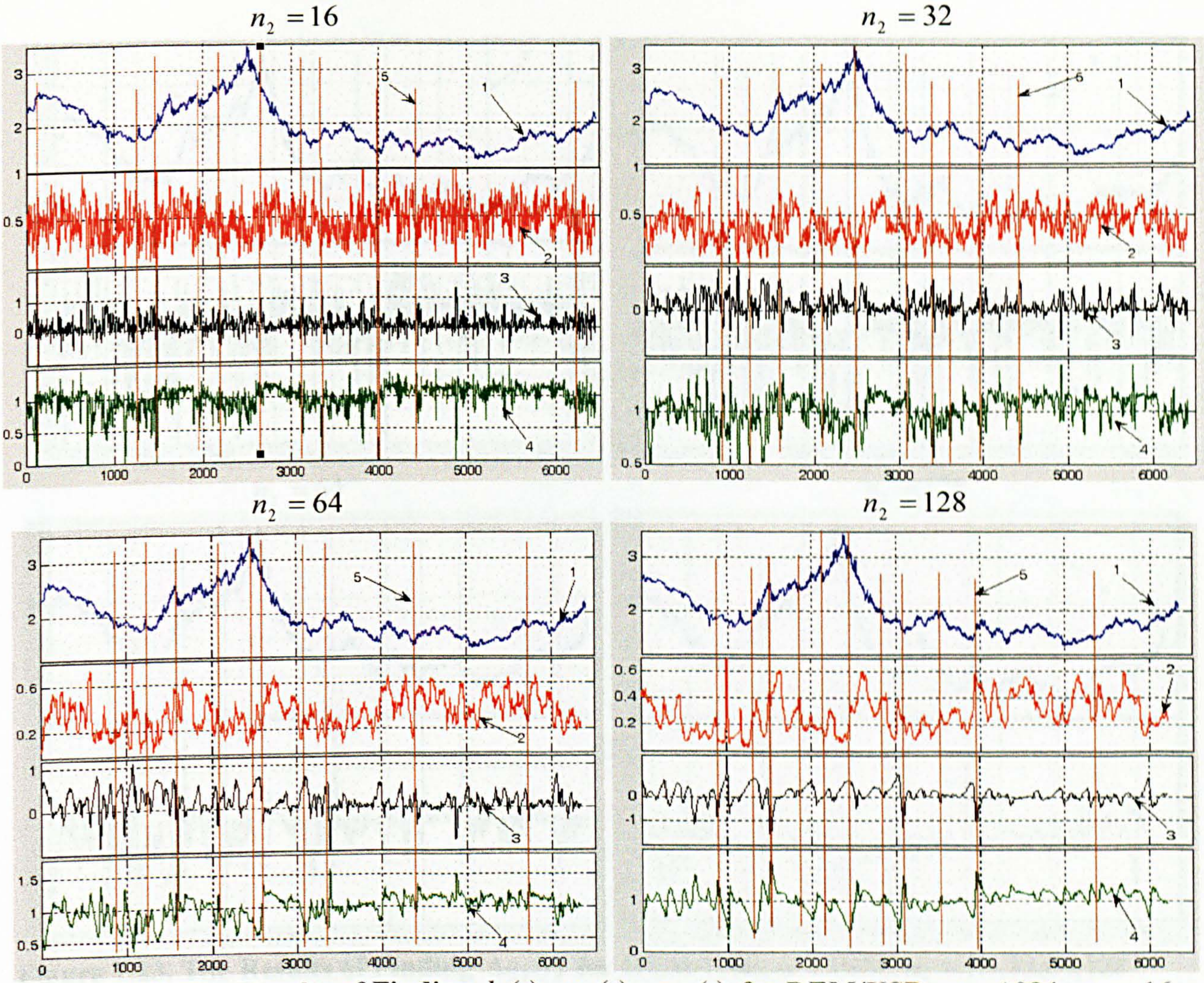


Figure 7.22. The Results of Finding $k_i(t)$, $q_{ie}(t)$, $q_{iec}(t)$ for DEM/USD, $m = 1024$, $n_2 = 16$, 32, 64, 128

1: initial quotation; **2:** $k_i(t)$; **3:** $q_{ie}(t)$; **4:** $q_{iec}(t)$; **5:** synchronised points of time

As a result, we get an opportunity for not only finding experimentally parameters $k_i(t)$ and $q_{iec}(t)$ at different points of time (with specific common sense), but also the prospects for more precise characterising of arbitrage attitudes of market participants in terms of $q_{ie}(t)$. In particular, an extra parameter $q_{ie}(t)$ is capable of characterising the current equilibrium value of $q(t)$, acceptable for all market participants. In other words, parameter $q_{ie}(t)$ can characterise the present arbitrage attitudes of market participants, e.g.: the degree of commitment in keeping existing tendencies (as $q_{ie}(t)$ grows), or changing ones (as $q_{ie}(t)$ falls). The parameters $\Delta q_2(t) = q_{iec}(t) - q_{ie}(t)$ can be used for characterising the degree of disagreements, uncertainty arrears, the possibility of crises at any point of time, etc. (Figure 7.23).

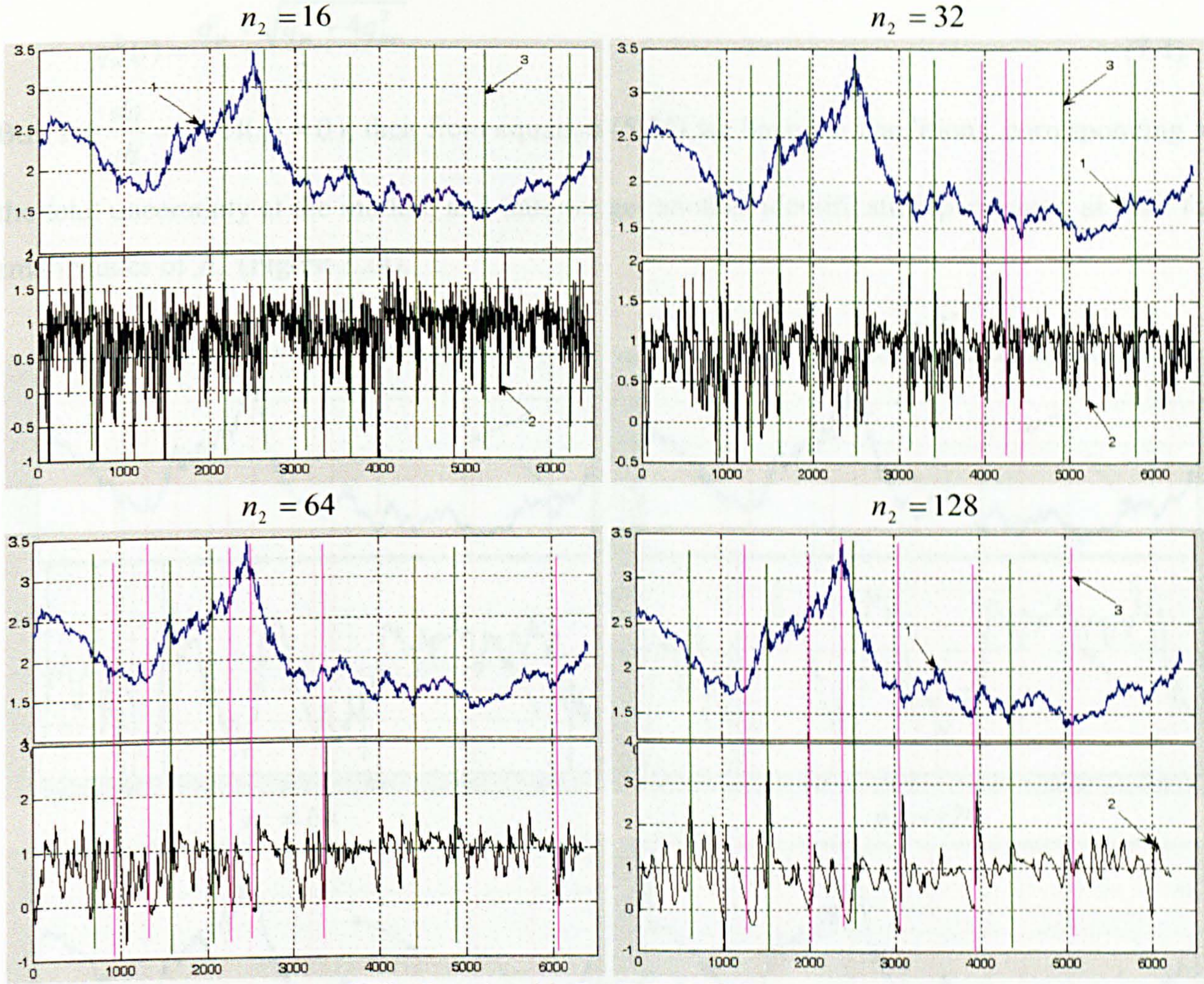


Figure 7.23. The Results of Finding $\Delta q_2(t)$ for DEM/USD, $m = 1024$, $n_2 = 16, 32, 64, 128$

1: initial quotation; **2:** $\Delta q_2(t)$; **3:** synchronised points of time

Results in Figures 7.22 and 7.23 unambiguously indicate the competence of the earlier assumptions on stabilising the impact of the control centres, and on the commitment of market participants both in changing and keeping the existing tendencies. These results support the conclusion on the growth over time of the impact by the control centres, since in these cases parameters $q_{iec}(t)$ and $\Delta q_2(t)$ are (usually) greater than zero and are close to unity. Thus, there are new perspectives in analysing the key reasons for a crisis situation to emerge, and, consequently, for their classification. Furthermore, there appears an opportunity for the complex behaviour estimation of single groups of market participants, and for analysing opinions of this part of the market at various points of time. For instance, we consider that the present equilibrium value $q^{\Sigma}_{ie}(t)$ for the whole market exists for $\frac{dq}{dt} = 0$, and then from equation (7.3) for $q > 0$, $q_{ie} \neq 0$, $k_i \neq 0$ we get:

$$q_{ie}^{\Sigma}(t) = \frac{q_{ie} + \sqrt{q_{ie}^2 + 4q_{iec}^2}}{2}. \tag{7.4}$$

But, for $\frac{dq}{dt} = 0$ ($A(q) = 0$), then from equation (5.24) we have the conditions, corresponding to the total uncertainty at the market, and thus we get another identification parameter, at least for small values of n_2 (Figure 7.24).

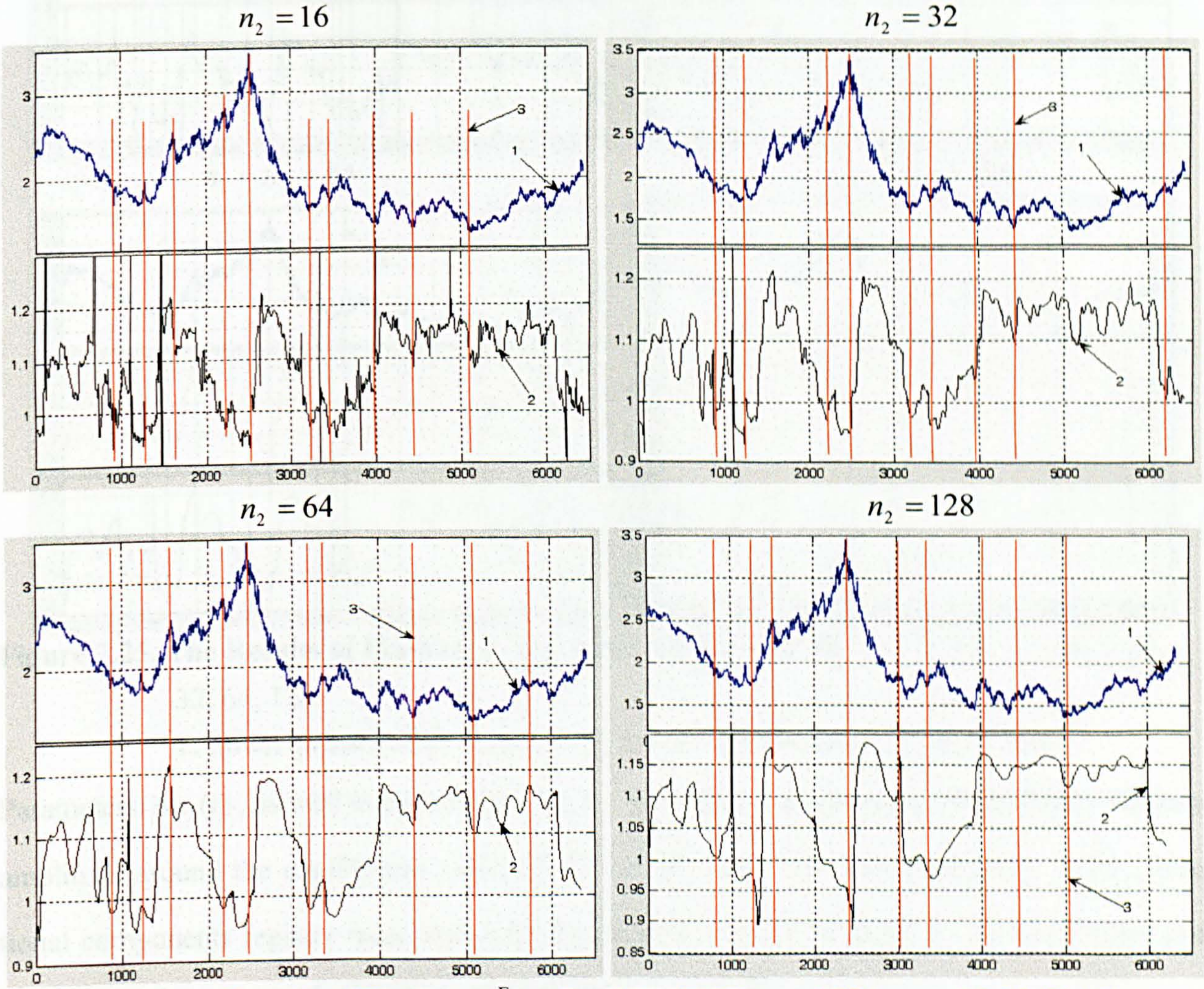


Figure 7.24. The Results of Finding $q_{ie}^{\Sigma}(t)$ for DEM/USD, $m = 1024, n_2 = 16, 32, 64, 128$

1: initial quotation; **2:** $q_{ie}^{\Sigma}(t)$; **3:** synchronised points of time

Taking into account that parameters $q_{ie}^{\Sigma}(t)$ and $q_{ec}(t)$ characterise the moments of crises' emergence in the same way, at this stage of the research it becomes possible to estimate time delays more precisely for different values of n_2 (Figure 7.25).

Together with estimation of parameters $q(t)$ and $q_{ie}^{\Sigma}(t)$, there is a possibility of finding parameters $\Delta q_3(t) = q(t) - q_{ie}^{\Sigma}(t)$, characterising the deviation of the system from the present equilibrium state $q_{ie}^{\Sigma}(t)$ (Figure 7.26).

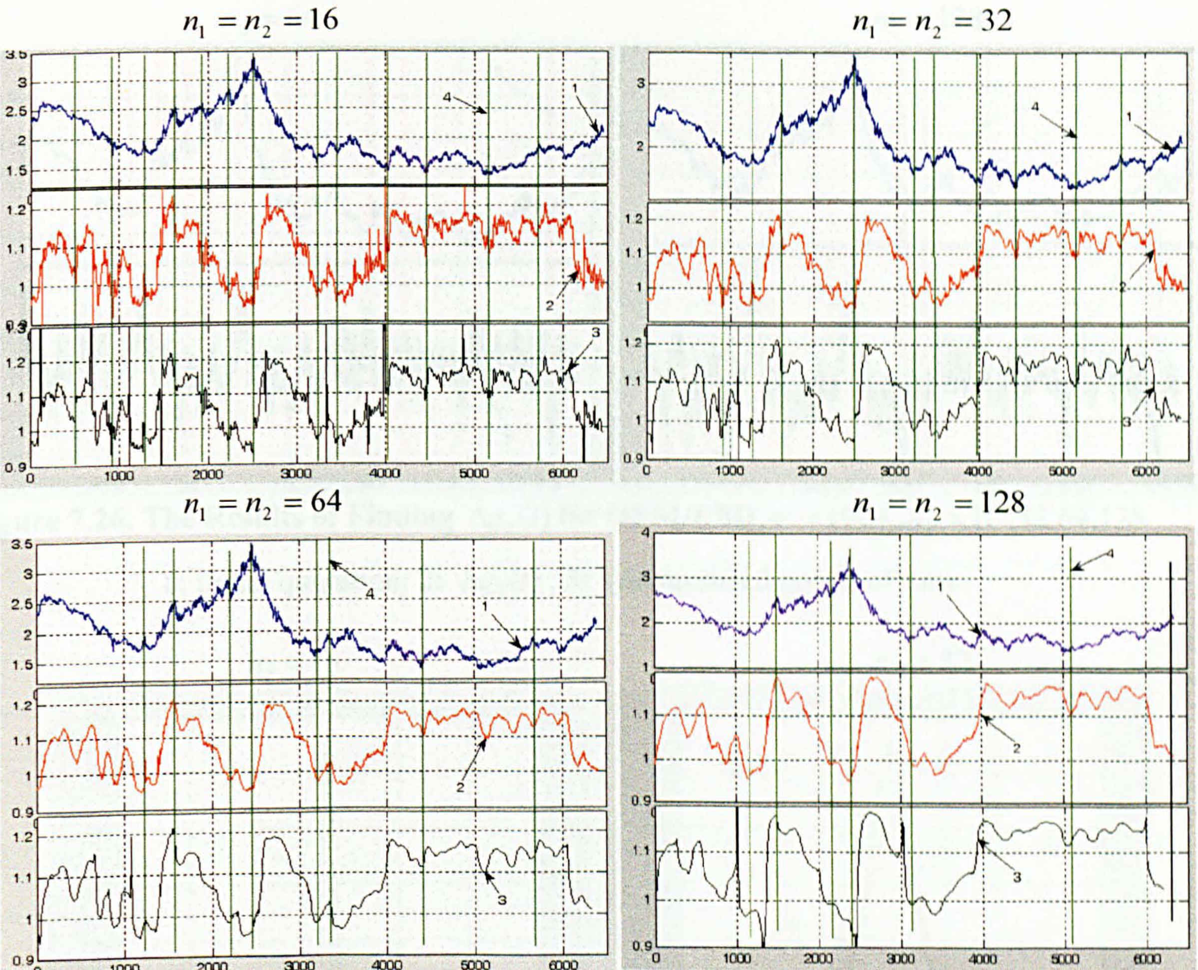
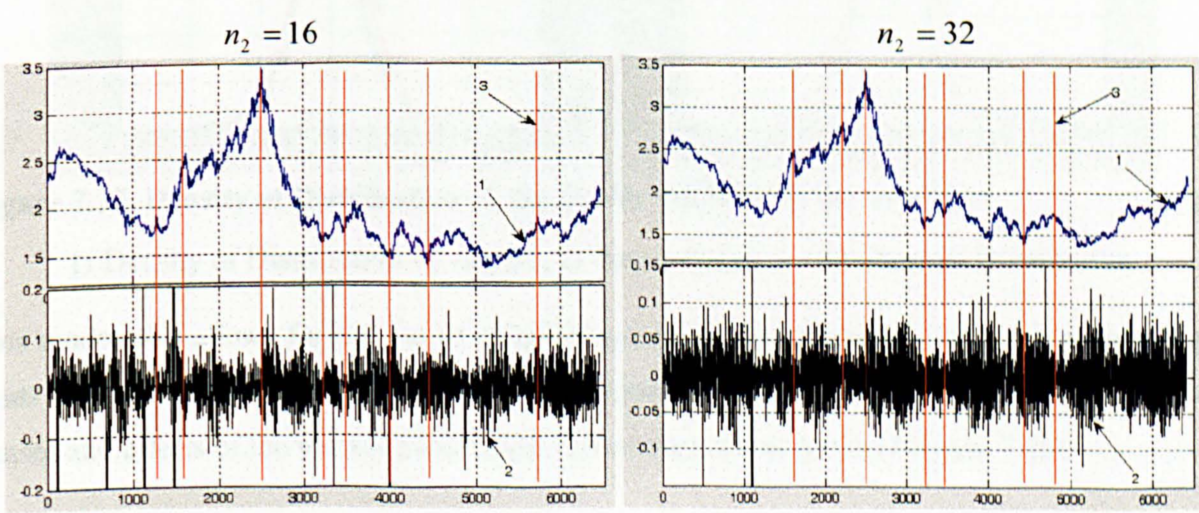


Figure 7.25. The Results of Finding $q_{ec}(t)$, $q_{ie}^{\Sigma}(t)$ **for DEM/USD,** $m = 1024$, $n_1 = n_2 = 16$, 32, 64, 128

1: initial quotation; **2:** $q_{ec}(t)$; **3:** $q_{ie}^{\Sigma}(t)$; **4:** synchronised points of time

Parameters $\Delta q_3(t)$, as well as parameters $\Delta q_1(t)$, are quite homogeneous, fluctuating with some amplitude around the equilibrium value $q_{ie}^{\Sigma}(t)$. At the same time, for parameters $\Delta q_3(t)$, fractional components register more distinctly for small n_2, while for high n_2 we get a more pronounced tendency towards distributions of the kind (5.39) (Figure 7.27).



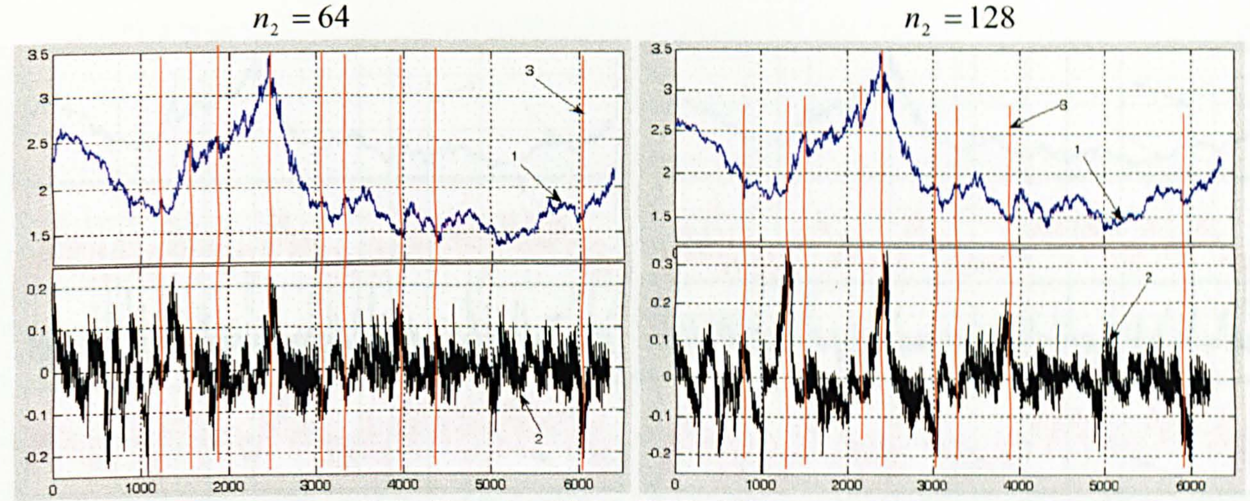


Figure 7.26. The Results of Finding $\Delta q_3(t)$ for DEM/USD, $m = 1024$, $n_2 = 16, 32, 64, 128$

1: initial quotation; **2:** $\Delta q_3(t)$; **3:** synchronised points of time

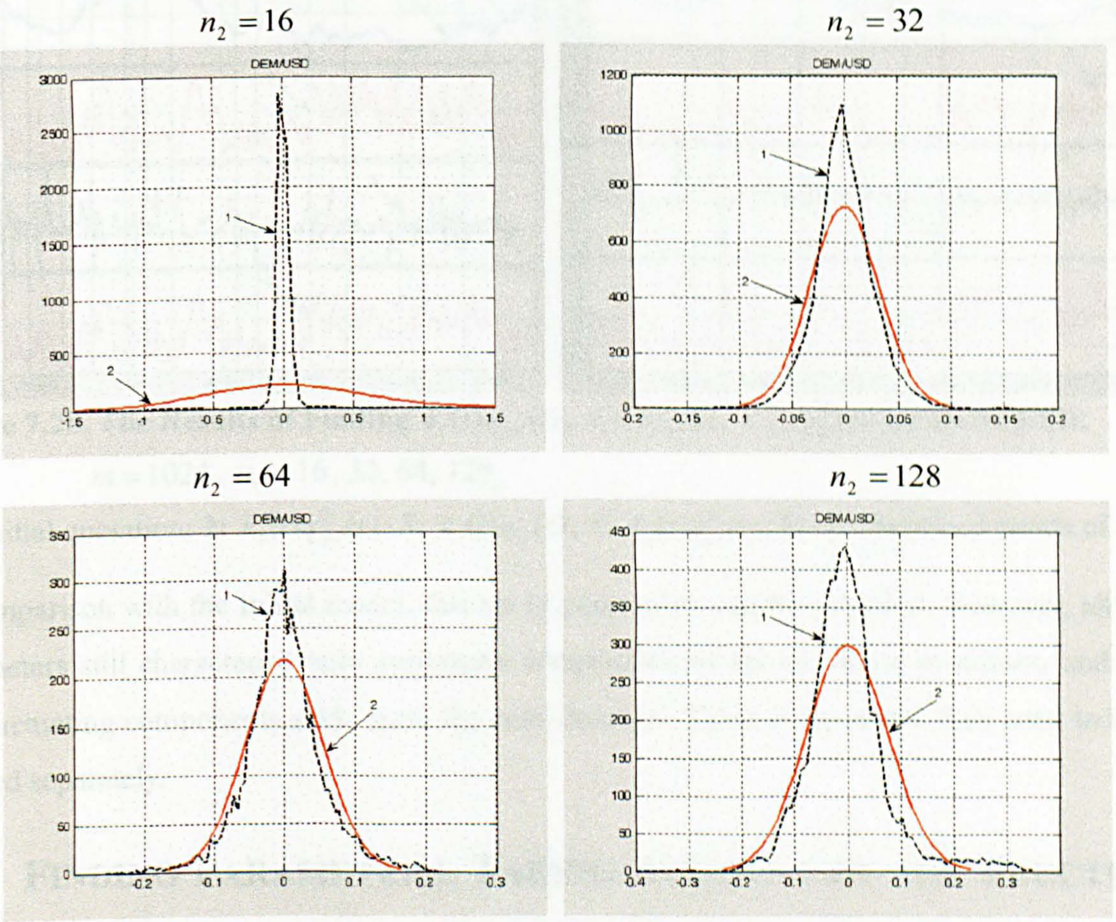


Figure 7.27. Density of Distribution of $\Delta q_3(t)$ for DEM/USD for $m = 1024$

1: Density of Distribution of $\Delta q_3(t)$; **2:** Approximation with Normal Distribution

This model also allows for the use of complex parameters for characterising systematic components of the occurring processes, and the interpretation of the obtained results in terms of expenses and efforts of the market participants and of the control centres (Figure 7.28).

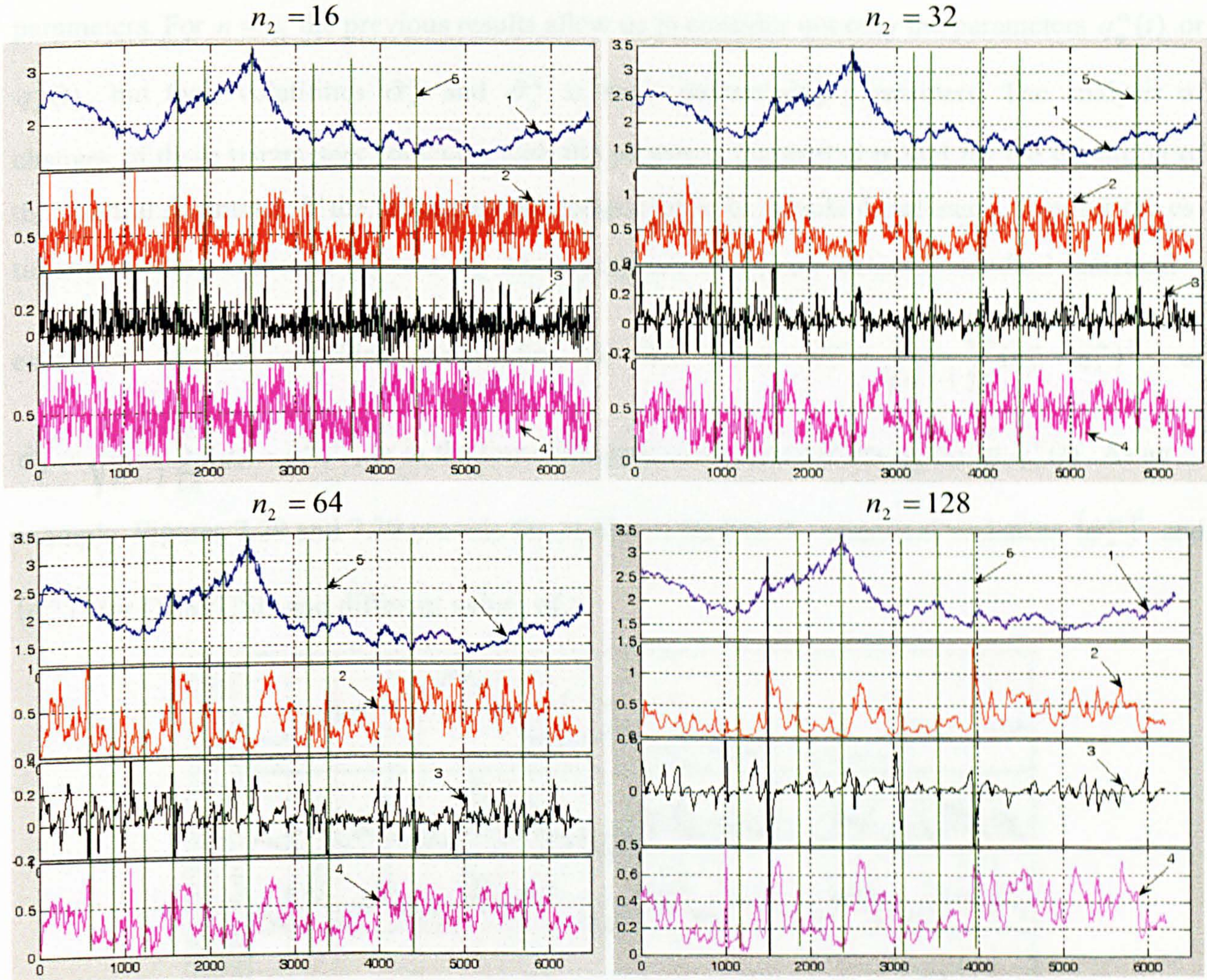


Figure 7.28. The Results of Finding $k_i(t)q^2_{iec}(t)$**,** $k_i(t)q_{ie}(t)$**,** $k_i(t)q^{\Sigma}_{ie}(t)$ **for DEM/USD,** $m = 1024$**,** $n_1 = 16, 32, 64, 128$

1: initial quotation; **2:** $k_i(t)q^2_{iec}(t)$; **3:** $k_i(t)q_{ie}(t)$; **4:** $k_i(t)q^{\Sigma}_{ie}(t)$; **5:** synchronised points of time

In comparison with the initial model, this set of parameters can be extended. However, all these parameters still characterise only systematic components of the occurring processes, and leave the fluctuating components aside from the consideration. These components then need to be examined separately.

7.5 Finding Parameters, Taking Account of the Fluctuating Components of the Process of the Evolution of the Spectral Parameter

When considering the fluctuating components, the empirical volatilities $\hat{\sigma}^m_n$ and $\hat{\sigma}^{\Sigma}_n$, obtained from n sequential values of $q^m_q(t)$ or $q^{\Sigma}_q(t)$, have to be considered among the main determining parameters. For $n = n_1$ the previous results allow us to consider not only the parameters $q_q^m(t)$ or $q_q^\Sigma(t)$, but their volatilities $\hat{\sigma}_n^m$ and $\hat{\sigma}_n^\Sigma$ as main determining parameters. The analysis of changes of these parameters coincides with the proposed theoretical model for the evolution of the spectral parameter. If the sequences are considered to be admitted stationary, then for the estimation of the parameters $\hat{\sigma}_n^m$ and $\hat{\sigma}_n^\Sigma$ it is appropriate to use (for instance) standard statistical estimates of the empirical volatilities in the form $\hat{\sigma}_n^m = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(q_k^m - \bar{q}_n^m)^2}$ or $\hat{\sigma}_n^\Sigma = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(q_k^\Sigma - \bar{q}_n^\Sigma)^2}$, or in the form of logarithms of parameters $q_q^m(t)$ or $q_q^\Sigma(t)$. As an example, Figures 7.29 and 7.30 provide the results of finding the empirical variances $(\hat{\sigma}_n^m)^2$ and $(\hat{\sigma}_n^\Sigma)^2$ for DEM/USD and different values of n.

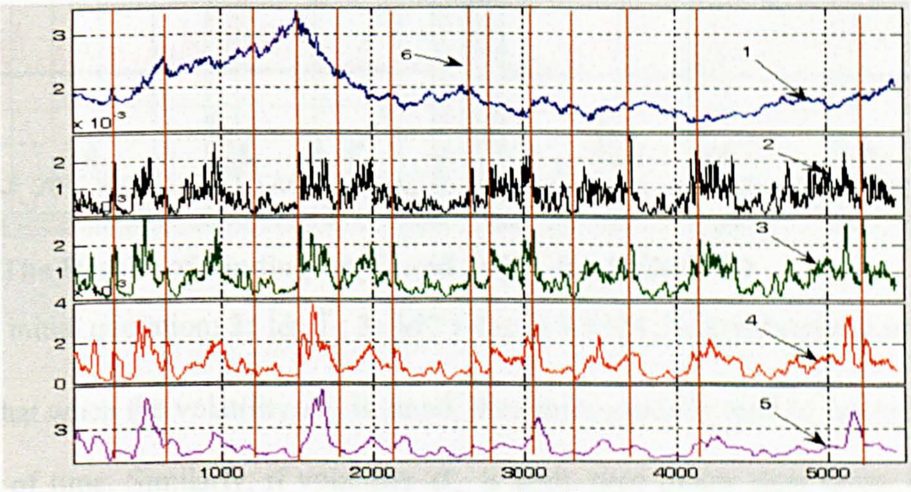


Figure 7.29. The Results of Finding the Variance $(\hat{\sigma}_n^m)^2$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** synchronised points of time

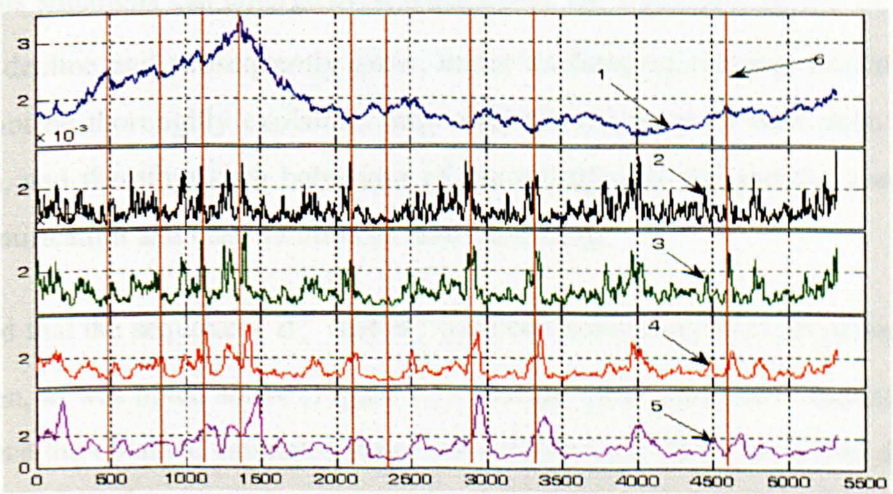


Figure 7.30. The Results of Finding the Variance $(\hat{\sigma}_n^\Sigma)^2$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** synchronised points of time

Sequences $(\hat{\sigma}_n^m)^2$ and $(\hat{\sigma}_n^\Sigma)^2$, as parameters characterise the intensity of the systematic components of the process, at least at the points of local extremums, exhibit some identification properties, especially when values of $(\hat{\sigma}_n^m)^2$ and $(\hat{\sigma}_n^\Sigma)^2$ change quickly. Of most importance is the fact, that volatilities $\hat{\sigma}_n^m$ and $\hat{\sigma}_n^\Sigma$, as well as volatilities of most of the financial data, are themselves volatile. In the behaviour of the parameters $\hat{\sigma}_n^m$ and $\hat{\sigma}_n^\Sigma$ there are some peculiarities, compared to the volatility $\hat{\sigma}_n$ of the financial data. (Figure 7.31).

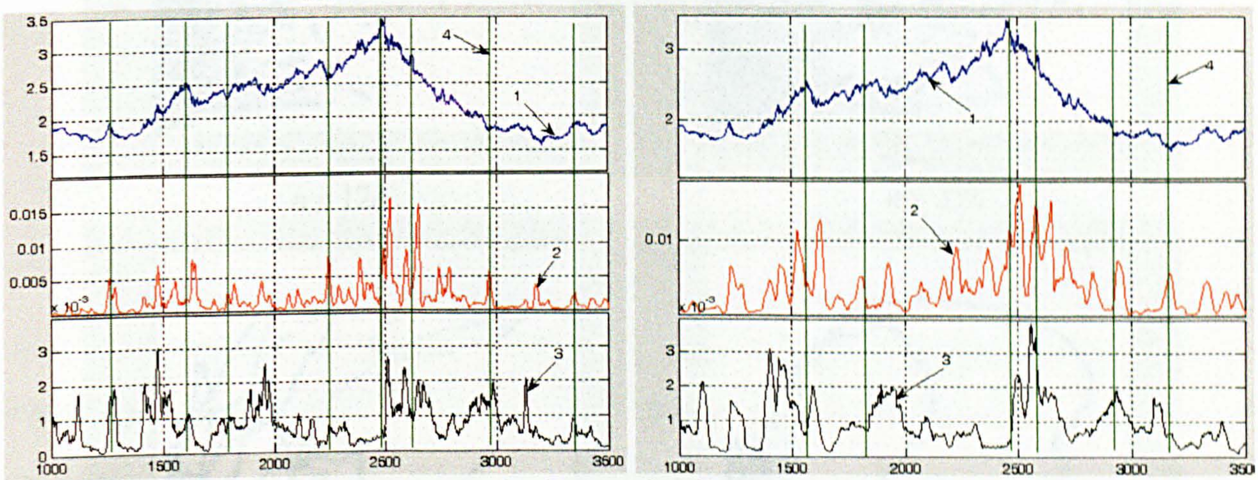


Figure 7.31. The Results of Finding $(\hat{\sigma}_n)^2$ and $(\hat{\sigma}_n^m)^2$ for DEM/USD

1: initial quotation; **2:** $(\hat{\sigma}_n)^2$; **3:** $(\hat{\sigma}_n^m)^2$ for $m = 1024$; **4:** synchronised points of time

It is known, that when the volatility $\hat{\sigma}_n$ is small, then prices usually tend to fall or rise for a sustained period of time. Similarly, if volatility $\hat{\sigma}_n$ is high, then prices slow down in fall or rise, tending to inverse price development. It follows from the presented results (Figures 7.30 and 7.31), that crisis situations can emerge even when either the volatilities of $\hat{\sigma}_n^m$ or $\hat{\sigma}_n^\Sigma$ are small, or when they decline and subsequently grow, under unchanging existing tendencies in prices. This fact cannot be thoroughly explained only with the existence of time delay in the results (Figures 7.32), and therefore such behaviour of the volatility of the spectral parameter can be used as an identification and classification of crisis situations.

If it is assumed that the sequences $\hat{\sigma}_n^m$ and $\hat{\sigma}_n^\Sigma$ maintain some important properties of the initial quotations, then, as was noted above (Figure 2.3), intraday heterogeneity of the initial data is unable to influence the obtained estimates, since we initially used daily quotations data. Thus, to a large extent the results have to characterise daily cycles (Figures 2.4 and 2.5), and estimates of parameters $\hat{\sigma}_n^m(t)$ and $\hat{\sigma}_n^\Sigma(t)$ have to be regular enough in time. Moreover, they still have to exhibit not only some fractional and identification properties, typical of the determining characteristics of the initial quotations and parameters $q_q^m(t)$ or $q_q^{\Sigma}(t)$, but also a number of phenomena, typical of non-linear systems' behaviour.

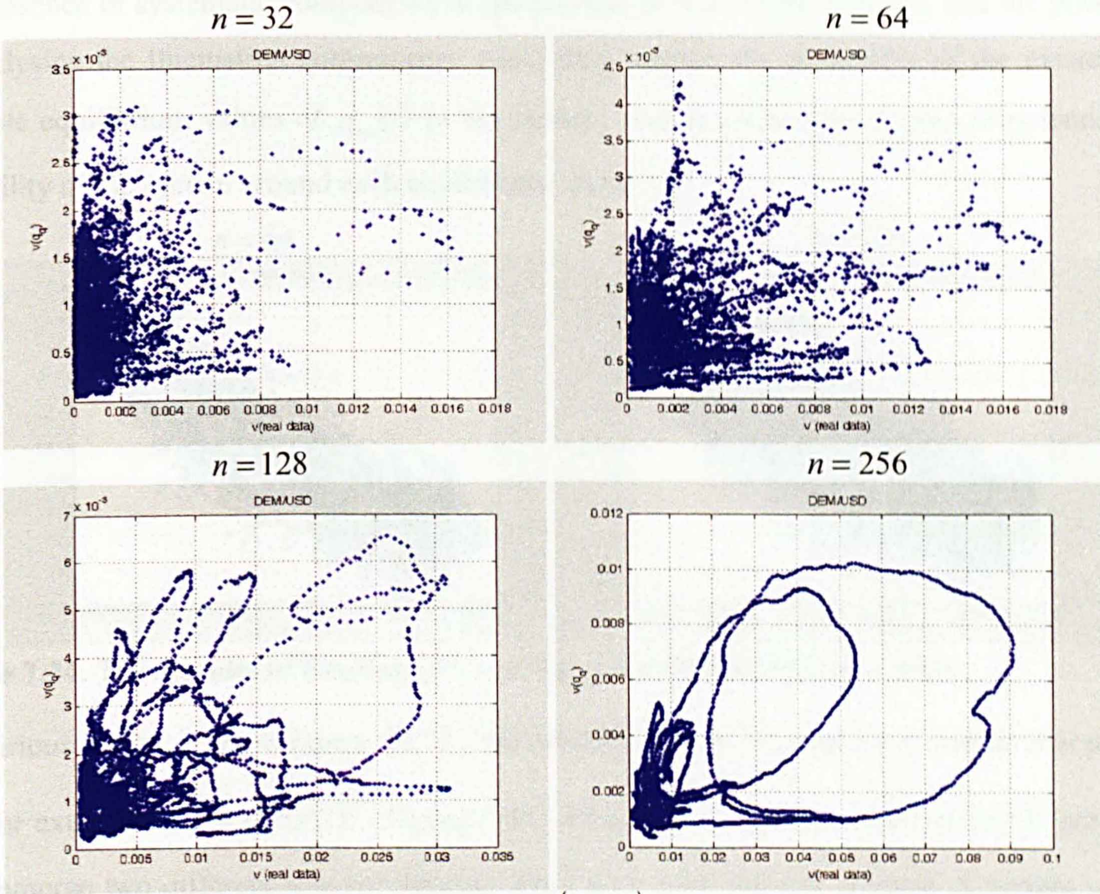


Figure 7.32. The Results of Finding $\left(\hat{\sigma}_n^m\right)^2 = f\left(\left(\hat{\sigma}_n\right)^2\right)$ for DEM/USD, $m = 1024$

The results indicate (Figure 7.33 for $N > 3000$) that for high *n*, parameters $\hat{\sigma}_n^m(t)$, together with parameters $q_{ec}(t)$ and $\rho^m(t)$, can be used for identification purposes.

Figure 7.33. The Results of Finding $\left(\hat{\sigma}_n^m\right)^2$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $\left(\hat{\sigma}_n^m\right)^2$ **for** $n = 1024$

In turn, this suggests that there is some statistical relationship among these parameters, which becomes more distinctive with increasing n (Figures 7.34 – 7.36). Figure 7.34 shows that extraneous fields of the second determining area almost became circles, which indicates the nearly total absence of systematic components of the process in the considered data, and the possibility of analysing the fluctuating components. Also, they support the possibility of the existence of multiple equilibrium values of $q_{ec}(t)$ in the system, and determine the degree of potential predictability of the system around each equilibrium value.

$n = 64$ $\qquad$ $n = 128$

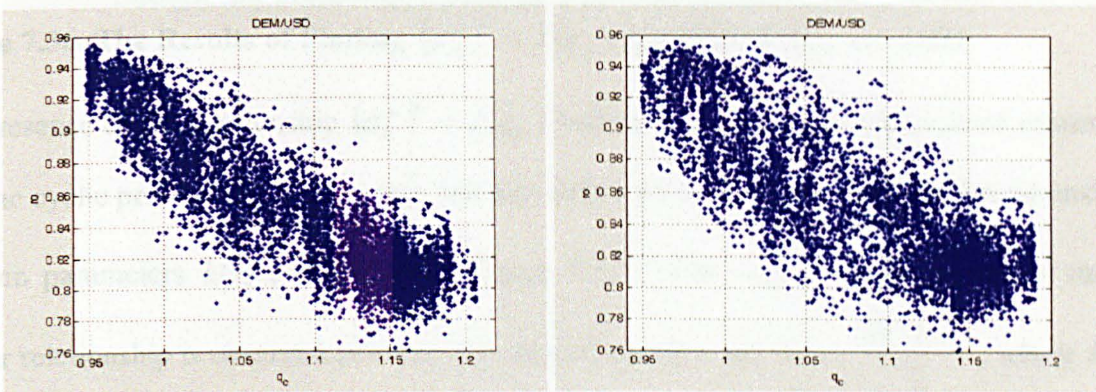


Figure 7.34. The Results of Finding $\rho^m = \rho^m(q_{ec})$ for DEM/USD, $m = 1024$

For various values of the variance $(\hat{\sigma}_n^m)^2$, the predictability of the system is varies. It is predictable for extreme values of $(\hat{\sigma}_n^m)^2$ (Figure 7.35). However, when the values of $(\hat{\sigma}_n^m)^2$ are small, there emerge two different non-overlapping areas with high and low degrees of system predictability.

$n = 64$ $\qquad$ $n = 128$

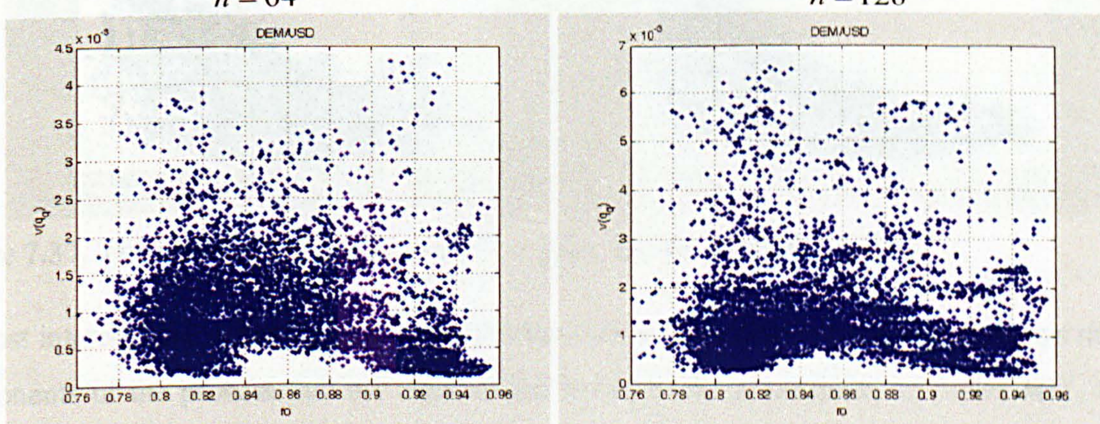


Figure 7.35. The Results of Finding $(\hat{\sigma}_n^m)^2 = f(\rho^m)$ for DEM/USD, $m = 1024$

When analysing the possible relationship between equilibrium values of $q_{ec}(t)$ and the parameters $\hat{\sigma}_n^m(t)$, the results in Figure 7.36 are seen to be similar to the behaviour of the parameters $k(t) = f(q_{ec}^2(t))$ (Figure 7.17).

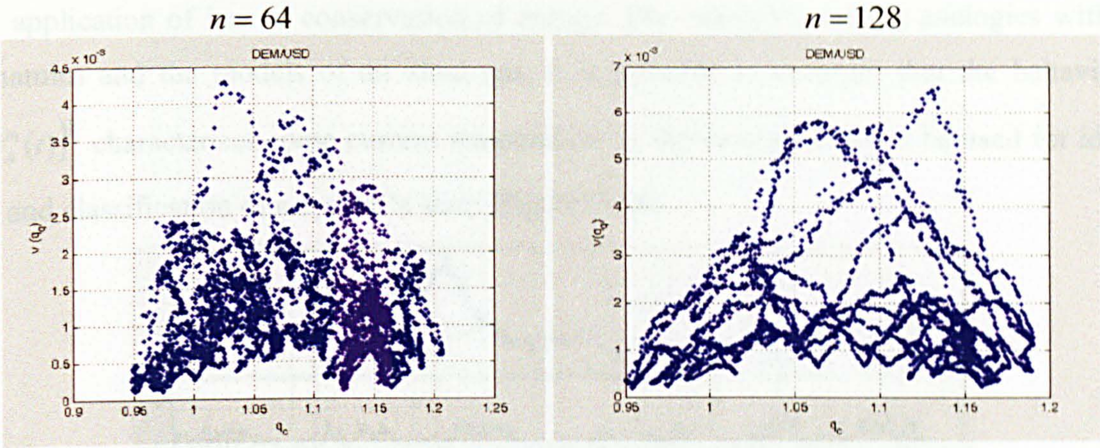


Figure 7.36. The Results of Finding $\left(\hat{\sigma}_n^m\right)^2 = f(q_{ec})$ for DEM/USD, $m = 1024$

The presence of the relationship $\left(\hat{\sigma}_n^m\right)^2 = f(q_{ec})$ indicates not only the existence of return points in some cyclic process $\hat{\sigma}_n^m = f(q_{ec}(t))$, but also determines the inverse statistical relationship between parameters $k(t)$ and $\left(\hat{\sigma}_n^m(t)\right)^2$ (Figure 7.37). With regards to the proposed model, a similar relationship is observed between the parameters $k(t)$ and $b(t) = \frac{\left(\hat{\sigma}_n^m(t)\right)^2}{2}$, where $k(t)$ characterises arbitrage activities (or costs) of the market participants, and $b(t)$ - the intensity of the random components of the process at different points of time.

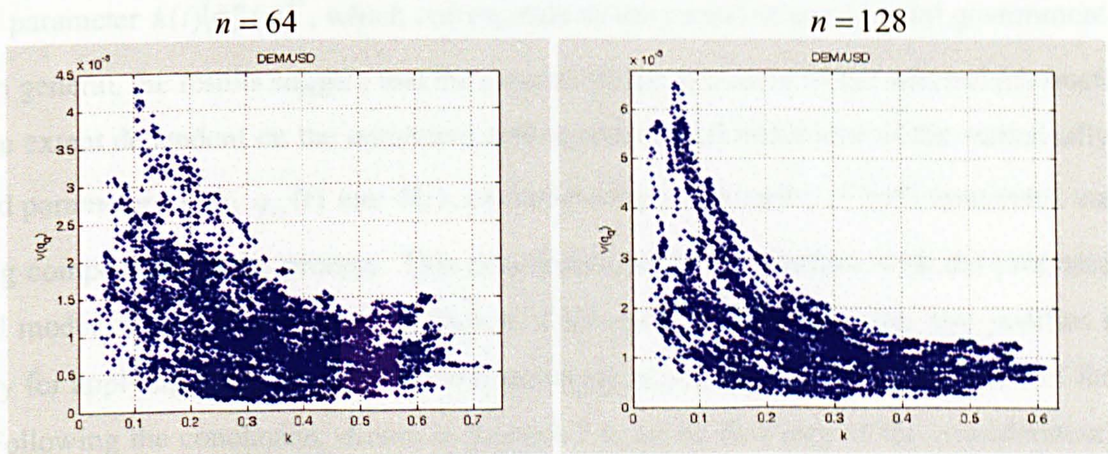


Figure 7.37. The Results of Finding $\left(\hat{\sigma}_n^m\right)^2 = f(k)$ for DEM/USD, $m = 1024$

Of most interest here is not the existence of a non-linear statistical relationship between different components of the process and the relationship between the parameters $k(t)$ and $b(t)$, but the inverse proportionality between parameters $k(t)$ and $b(t)$. In other words, it is possible to state that with the increasing arbitrage activities of the market participants, the intensity of random components of the process drops, and vice versa. It seems, that here we are dealing with self-organisation of the system, occurring at different levels in various components under some diffusion chaos. This behaviour is, for instance, typical of some closed systems, which makes possible an application of law of conservation of energy. For example, making analogies with thermodynamics and the models of an ideal gas, it is possible to consider that the behaviour of $k(t)\left(\hat{\sigma}_n^m(t)\right)^2$ characterises some current temperature of the system, and can be used for identification and classification of a system's state (Figure 7.38).

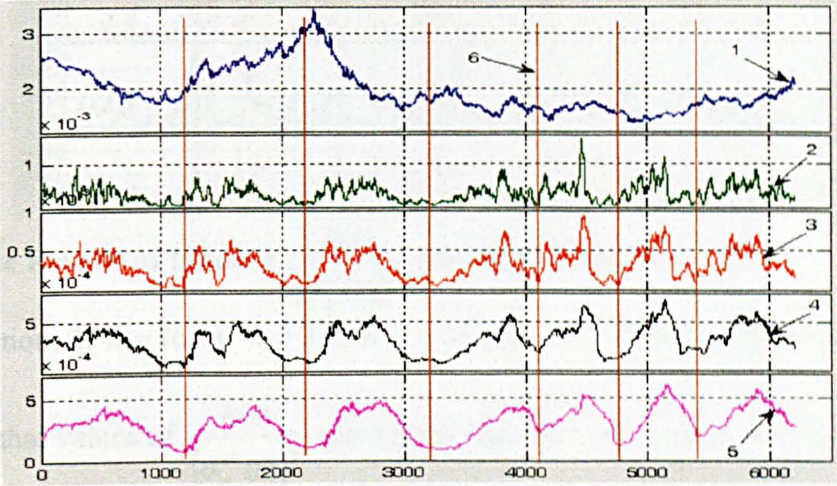


Figure 7.38. The Results of Finding $k(t)\left(\hat{\sigma}_n^m(t)\right)^2$ for DEM/USD, $m = 1024$
1: initial quotation; **2:** $n = 32$; **3:** $n = 64$; **4:** $n = 128$; **5:** $n = 256$; **6:** synchronised points of time

It is interesting to note, that there is a four-year regularity of cyclic recurrence in the behaviour of the parameter $k(t)\left(\hat{\sigma}_n^m(t)\right)^2$, which corresponds to the period of presidential government in the US. In general, the results suggest that the process of the evolution of the spectral parameter is to a large extent dependent on the combined self-organisational behaviour of the statistically interrelated parameters $k(t)$, $q_{ec}(t)$ and $b(t)$, characterising the intensity of both systematic and fluctuating components of the process. This conclusion not only coincides with the proposed theoretical model of the process of the evolution of the spectral parameter, but also justifies the necessity for applying complex criteria, simultaneously accounting for all components of the process. Following the conclusion, drawn in Section 5.4, let us first turn to the consideration of the ratio $\frac{k(t)}{2b(t)} = \frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$, which is a complex criterion, and, according to the proposed model, is included into the PDF (equation (5.36) and (5.39)), and characterises the relationship between arbitrage costs of the market participants and the intensity of random components of the process at different points of time (Figure 7.39). A sharp decline in $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ can act as the identifier of possible changes (innovations) occurring, while an increase in $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ can act as an identifier, predetermining (antecedent to) crisis situation (harbinger of innovations).

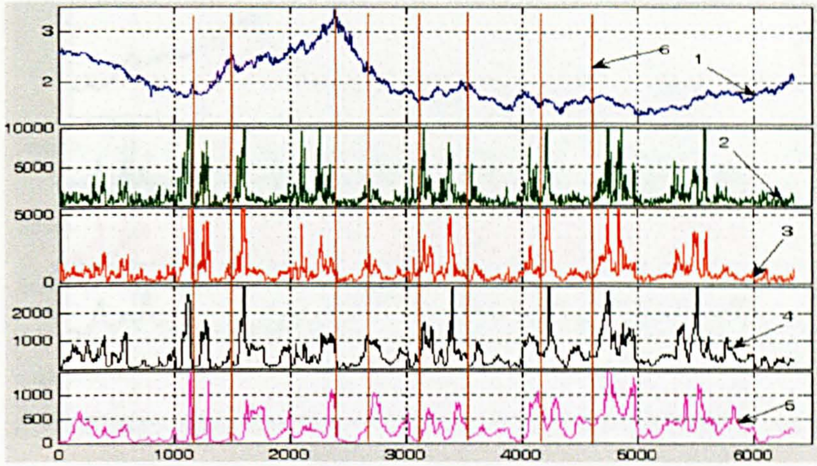


Figure 7.39. The Results of Finding $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** synchronised points of time

It is significant that values of $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ can not be high for a long time, as sooner or later every peak goes down. This implies that at extremum values of $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ correspond to the return points, mentioned above, and allows for non-trivial forecasting. The described behaviour of the $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ ratio barely resembles strong turbulence (at least for small n), when some chaotic behave with infrequent but extra-high outbursts can be observed in the dynamic system. At the same time, with regards to the considered ratio, it is possible to conclude that the dominating impact of the variance parameters $\left(\hat{\sigma}_n^m(t)\right)^2$, characterises the intensity of the random components of the process. Since $k(t) \sim \frac{c}{\left(\hat{\sigma}_n^m(t)\right)^n} \sim 0.2,...,0.9$; $n \sim 2$ (Figure 7.37) and $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2} \sim \frac{c}{\left(\hat{\sigma}_n^m(t)\right)^n} \sim 100,..,10000$; $n \sim 4$ then changes in parameters $k(t)$ are unable to change the order of $\frac{k(t)}{\left(\hat{\sigma}_n^m(t)\right)^2}$ (Figure 7.40). It has to be emphasised that the parameter $\frac{k(t)}{2b(t)}$ is included into not only the distribution function (5.39) as a multiplier $\exp\left(-\frac{kq^2}{2b}\right) = \exp\left(-\frac{k(t)q^2(t)}{\hat{\sigma}^2(t)}\right)$, which to a large extent determines the behaviour of this function to the right of the mode, but also into a similar multiplier $\exp\left(-\frac{q^2}{2\hat{\sigma}^2(t)}\right)$, which contains a normal distribution. For $\Theta >> 1$ (equation (5.39)), both of these distributions are δ-shaped limiting distributions, which almost coincide only for $k(t) \approx 0.25$ (Figure 7.41).

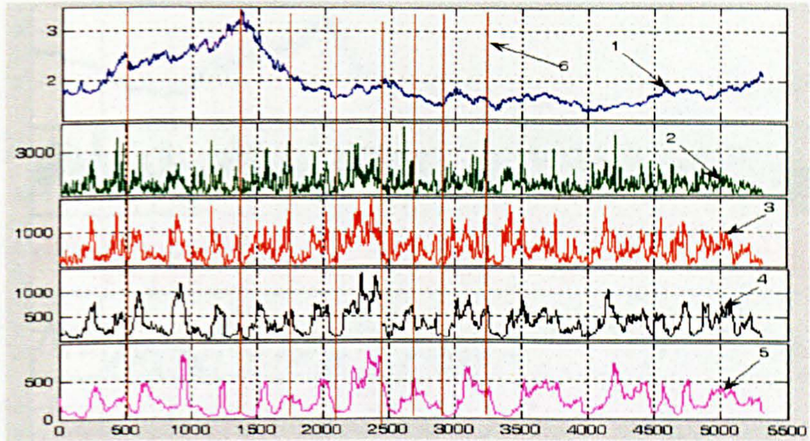


Figure 7.40. The Results of Finding $\frac{k(t)}{\left(\hat{\sigma}_n^{\Sigma}(t)\right)^2}$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** synchronised points of time

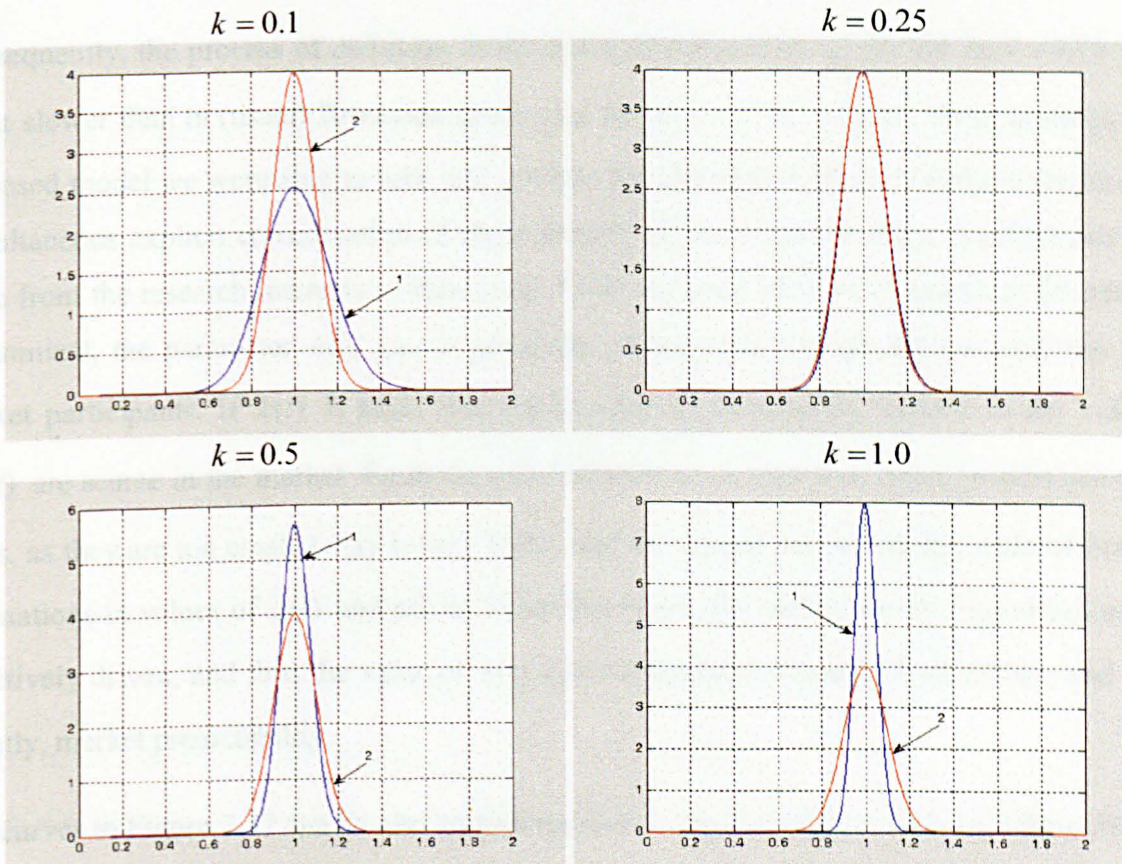


Figure 7.41. Dependence of Normal and Limiting Distribution $f(q)$ (5.39), for $\Theta >> 1$

1: The PDF of $f(q)$ (5.39); **2:** The PDF of Normal Distribution

Then, the scope of deviation from normality with the same variance $\hat{\sigma}^2(t)$ can be approximated with the difference $k(t) - 0.25$ (Figure 7.42). When the differences $k(t) - 0.25$ are positive ($k(t) - 0.25 > 0$), distribution functions of parameters $q_q^m(t)$ decline faster than in the case of the normal distribution; and when the differences are negative ($k(t) - 0.25 < 0$) – they decline slower (Figures 7.41 and 7.42).

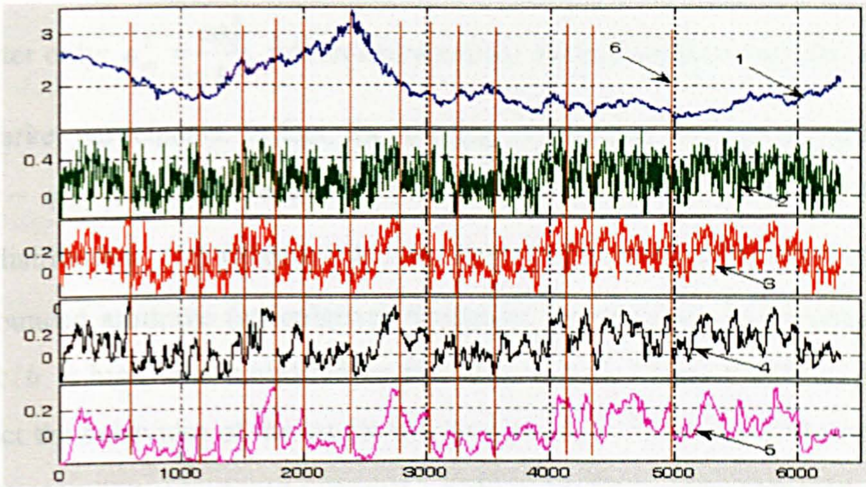


Figure 7.42. The Results of Finding $k(t) - 0.25$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** synchronised points of time

Consequently, the process of diffusion in the space of parameters $q_q^m(t)$ for $k(t) > 0.25$ has to occur slower than in (usual) Brownian space; and for $k(t) < 0.25$ – faster. Thus, in terms of the proposed model we were able to take into account the oblongness of the empirical distributions. Simultaneous explicit consideration of the existence of heavy tails in these distributions is left aside from the research interests of this study. From the point of view of practical financial and economical, the parameter $k(t)$ can in particular characterise the speculative activities of the market participants. If $k(t)$ is high, this implies that an anomalous outburst in the values of $q_q^m(t)$ are scarce in the market. From the point of view of energy loss, these outliers provide no gains, as they are too costly ($k(t)$ is very high), and the market acts smoothly, without extensive fluctuations in values of $q(t)$ and prices. From this point, the market can be considered as more effectively driven, and thus the value of $k(t)$ may represent a measure of efficiency, and subsequently, market predictability.

The curves in Figure 7.37 can be also considered as the classical demand curves, where $k(t)$ acts as some price, which has to be paid for possible changes $q(t)$, and $\left(\hat{\sigma}_n^m(t)\right)^2$ is the number of market participants, ready to pay this price. It is evident that the lower this price is, the more market participants will be ready to pay it (having only speculative interests), and the more often these anomalous outbursts occur (outliers) $q(t)$, subsequently, a market crisis will occur. Thus, we have supported the possibility of using the parameters $k(t)$ as identifiers, and also found some new meaning for this parameter, including its application for changes in fluctuating components of the processes. Also for $\Theta >> 1$ it could be considered that $f(t)$ is dependent basically on one parameter only: $q_{ec}^2 = \frac{cq_0^2}{k}$, which characterises the activities of the CBs in relation to the activities of market participants. In fact, the limiting distribution (5.39) is determined with only one parameter – q_{ec}, and an increase in k (within some intervals) can result in only a minor change in the distribution $f(t)$ (Figure 7.40). In other words, under effective centralised governance, all pronounced arbitrage (speculative) tendencies in activities of the market participants ($k >> b$ and k/b is high) are controlled by the CBs, and in a majority of cases, are unable to seriously impact the behaviour of the whole market ($q \approx q_{ec}$). In this case, the emergence of crisis situations for the market is determined only with anomalous external factors, and/or ineffective activities of the CBs, i.e. with current values of parameters q_{ec}. This justifies the possibility of using parameters $\Theta = \frac{kq_{ec}^2}{b} = \frac{cq_0^2}{b}$ as identifiers, characterising activities of the CBs, aimed at maintaining the present equilibrium value of q_{ec} in relation to the intensity of fluctuating components of the processes in terms of the proposed model (Figure 7.43).

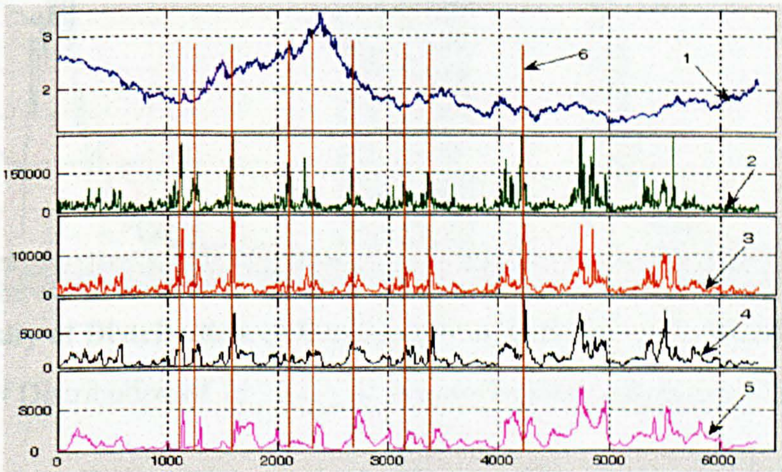


Figure 7.43. The Results of Finding $\Theta = \frac{kq_{ec}^2}{b}$ for DEM/USD, $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** synchronised points of time

The parameter $\Theta = \frac{kq_{ec}^2}{b}$ is included into the PDF of (5.39) in the form of a multiplier t^{Θ}, which (to large extent) characterises the behaviour of this function to the left of the mode, and has almost the same identification properties, that parameter k/b have, but in a more explicit form. However, despite the fact, that parameters $\Theta = \frac{kq_{ec}^2}{b}$ and k/b characterise relationships among various components of the process, but have different meanings, for $\Theta >> 1$, and for identification purposes, these parameters have to be considered equivalent. At the same time, for both parameters, volatility (empirical volatility), whose PDF for the complete sequence is very far from being normally distributed, can still be considered as dominant in these parameters (Figure 7.44). From parameters $\left(\hat{\sigma}_n^m(t)\right)^2$, similar conclusions can be drawn for other parameters of the modifications of the proposed model.

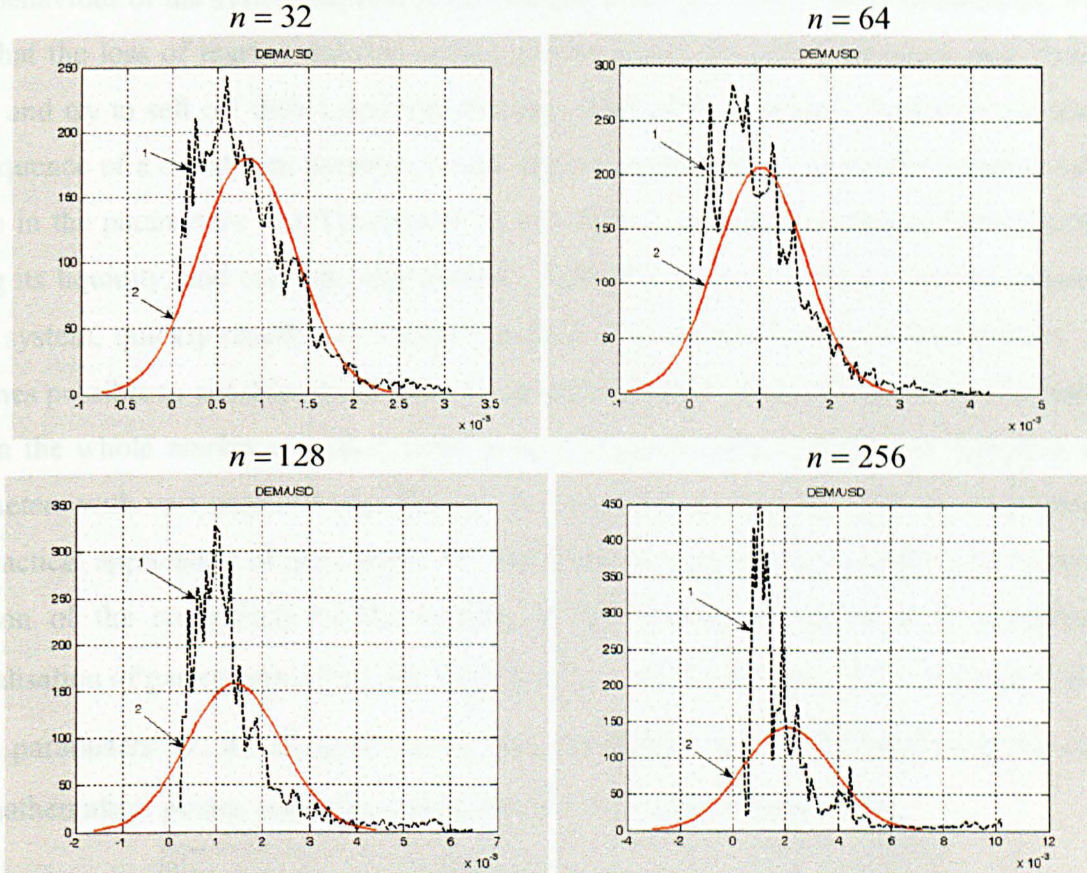


Figure 7.44. Density of Distribution of Empirical Variance $\left(\hat{\sigma}_n^m(t)\right)^2$, DEM/USD, $m = 1024$

1: Density of Distribution of $\left(\hat{\sigma}_n^m(t)\right)^2$; **2:** Approximation with Normal Distribution

In conclusion, let us consider one more possible meaning of the parameter $\Theta = \frac{kq_{ec}^2}{b}$. For $\Theta > 10$, for the given accuracy it is possible to consider Θ as a natural number. Then, having defined $\Theta = l - 1$ and $\hat{\sigma}_l^2 = \frac{b}{k}$, for $0 < q < \infty$ from equations (5.39) and (5.40) we get: $f(q) = 2q^{l-1} \exp \frac{(-q^2)/(2\hat{\sigma}_l^2)}{\left(2\hat{\sigma}_l^2\right)^{\frac{l}{2}} \Gamma\left(\frac{l}{2}\right)}$. This ratio is a PDF of the module of the multivariate vector with exactly l components (independent, random and normally distributed). Thus, it can be considered that with the integer value $\Theta = l - 1$ it is actually possible to describe the complexity of the system and the number of its degrees of freedom in the form $l = \Theta + 1$. In other words, with parameter $\Theta(t)$ we are not only able to describe the number of market participants (the number of groups of market participants) with their own investment horizons, but are also able to control how this number is changing at different points of time. Therefore it becomes clear why for high values of m, the existence of crisis situations is usually accompanied by declines in the parameter Θ. These declines occur exactly at those points of time, when the processes of self-organisation took place in the system and the number of degrees of freedom in the system goes down sharply. This behaviour of the system experimentally supports the position of most researchers, who indicate that the loss of market stability occurs exactly when long-term investors stop their operations, and try to sell off their assets and become short-term investors. Similar result could be a consequence of a significant number of new market participants with similar interests (sharp increase in the parameters Θ). Parameters Θ are able to characterise market diversification, ensuring its liquidity, and consequently stability. From this point, FX-market can be considered as some system, quickly reacting to changes in time of its diversification characteristics. Thus, it becomes possible to consider parameters Θ as quite reliable statistical indicators of crises occurring in the whole market or within some groups of participants, and also to recognise them as parameters with vast practical significance. It is possible to continue with the consideration of the practical application of parameters Θ. For instance, they can be used for a more precise estimation of the complexity of the system, or for finding the order of its parameters, renormalisation of parameters $q_q^m(t)$ for finding $q_q^{\Sigma}(t)$, etc. At the same time, regardless of the use of the parameters Θ, the obtained results and conclusions need to be taken only for statistical and mathematical points, under the condition $10^{1.5} < \Theta < 10^4$ (Figure 7.45).

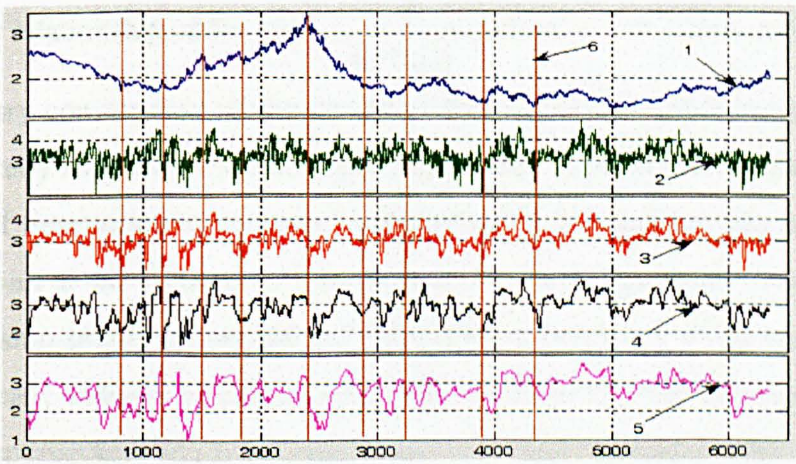


Figure 7.45. The Results of Finding $\lg\Theta = \lg\dfrac{kq_{ec}^2}{b}$ **for DEM/USD,** $m = 1024$

1: initial quotation; **2:** $n = 16$; **3:** $n = 32$; **4:** $n = 64$; **5:** $n = 128$; **6:** points of synchronisation

This, in turn, suggests that (at least) for identification and classification purposes, to a first approximation, the process of the evolution of the spectral parameter with a fair accuracy can be described with stochastic diffusive equations, including the FPK equation.

7.6 CONCLUSION

There can be no "correct single model" for the FX-market, and its description could represents a complex of systems with various aims and areas of development, among which there is no sense of looking for the "correct" state of the system, but the current state of the system can be observed. There can be quite a lot of local equilibria states of the FX-market, and in the long-term the contemporary FX-market can be considered as some control system, but controlled only by the CBs, what entirely coincides with the main characteristics of this market. Besides, values of the coefficient of pairwise correlation of the direct and the inverse regression of the spectral parameter can be considered not only as the identifiers of the determining areas of the system, but also as parameters, characterising the predictability of system's behaviour and the accuracy of possible future forecasting. For identification and classification purposes, to a first approximation, the process of the evolution of the spectral parameter can be described using stochastic diffusive equations, including the FPK equation. The use of the proposed methodology for experimentally finding the main quantitative parameters, determining systematic and fluctuating components of the process of the evolution of the spectral parameter, provides fair results. Here, the dominating impact of the empirical volatility of the process of evolution of the spectral parameter is supported together with the use of models of equilibrium dynamics for a description of changes in the systematic components of the process of the evolution of the spectral parameter. It is found that while using complex parameters of the process, the change of empirical volatility has inverse power relationship of the type $\sim \frac{c}{\left(\hat{\sigma}_n^m(t)\right)^n}$, where $n \sim 4$. These results not only indicate the satisfactory convergence of the results of theoretical and experimental analysis of this thesis, and practically support the possibility of application of methods of statistical fractionality for the analysis of time series of currency exchange rates; but also provide grounds for further research. They point at the presence of the processes of self-organisation and the existence of statistical relationship among systematic and fluctuating components of the process of evolution of spectral parameter, which is non-linear and cyclic in character. This stipulates the use of complex criteria and approaches, which not only allow us to find the determining characteristics and mechanisms of dynamics in the system, but also indicate possible future development lines. It is evident that if such problem could be solved without the correspondingly developed theory, but rather with relatively simple engineering methods, then this problem is easily solvable. All the obtained results and conclusions have to be treated in complex only, and have to be considered as statistical and probabilistic points, since the problems of financial analysis of the FX-market are quite numerous, and are too far from being theoretically and practically solved yet.

CHAPTER VIII. Discussion and Future Work

8.1 MAIN RESULTS AND SUMMARY OF THE RESEARCH

The following conclusion can be drawn from this research:

1. It was found, that errors emerging from a general form of the use of OLS for estimating the currency time series spectral parameter values leads to both erroneous quantitative estimates, and results in dramatic misinterpretations in the nature of the occurring processes, which suggests that OLS is ineffective in this case. At the same time, the use of the method of orthogonal regression for amplitude spectrums is justified for the identification problem.
2. It was experimentally obtained that with regards to currency quotations, the spectral parameter, included within a non-stationary fractional differential equation of time dependent order, can be used not only as a macroeconomic parameter, but also used for identification and classification purposes, and, sometimes, for forecasting.
3. The proposed method for finding the spectral parameter, which is based on the use of Fourier apparatus, and the use of orthogonal regression, allows for good quantitative estimates of the spectral parameter $q(t)$, and is favourably comparable with other methods of similar analysis (in particular to *R/S Analysis*).
4. With regards to mean values, the experimental results obtained suggest the existence of a relationship between the values of the spectral parameter and the Hurst parameter in the form $q_{R/S} = H + ½$ with ratio error of no more than 7%. This indicates the appropriateness of using the FBM-based models for handling the time series for currency exchange rates.
5. The proposed method for finding the spectral parameter can be considered appropriate for objective quantitative estimation of the mean values for the fractional dimension when considered time series for currency exchange rates. It was experimentally obtained that sequences of the spectral parameter, as well as sequences of the main quantitative characteristics of their evolution, maintain most of the important properties of the initial quotations, including the properties of non-linearity, fractionality and statistical probability.
6. It was verified that methods of detailed classification of the stochastic and chaotic components of the processes, established using the correlation integral, do not always pro-

vide unambiguous results when applied to time series for currency exchange rates. At the same time, these methods do indicate that the process of change of the spectral parameter is more stochastic than chaotic, and the sequence of the spectral parameter can be attributed to correlated random sequences with the correlation increasing together with the size of the reconstruction window. It is clear that this conclusion does not lead to the rejection of the hypothesis that other chaotic sequences with relatively small dimensions could have similar properties.

7. The relatively low correlation of the values of the spectral parameter suggests that, at least, to a first approximation, the use of stochastic equations for the simulation of the process of the evolution of the spectral parameter is justified. The possibility of using statistical methods for estimating the process of the evolution of the spectral parameter has been theoretically developed and empirically proved.
8. It was found that, at least, for identification and classification purposes the process of the evolution of the spectral parameter can be described with diffusive-type equations, including the FPK equation.
9. A model for the spectral parameter evolution was proposed. In terms of this model it was found that for the FX-market there could be no "correct single model", and its description could be made through various integrated systems with different aims and development areas, among which there is no sense of looking for the "correct" state of the system, but a present state can be observed. The possibility of the existence of many local equilibria of the FX-market in the general case has been experimentally supported. In the long-run the FX-market can be attributed to the controlled system, which is controlled and monitored by the CBs. It is indicated that models of equilibrium dynamics are entirely applicable for describing the systematic changes in the process of evolution of the spectral parameter.
10. It was found that, the values of the pairwise correlation coefficient of the direct and the inverse regressions of the spectral parameter can be considered not only as identifiers of the determining areas of the system, but also as parameters, characterising the degrees of the system's predictability and accuracy of forecasting.
11. In terms of the proposed theoretical model, the methodology for experimentation by finding the main quantitative parameters, determining systematic and fluctuating components of the process of evolution of the spectral parameter, including complex indices for the whole process, has been defined. The existence of a statistical relationship between systematic and fluctuating components has been experimentally supported. The

possibility of interpretation of these components in both physical and financial/economical terms has been provided.

12. The dominating impact of empirical volatility of the process of the spectral parameter evolution, whose PDF for the complete sequence is far being normally distributed, has been experimentally supported. It was found that while using complex indices for identification and classification, the change of empirical volatility is inverse proportional to $\frac{c}{\left(\hat{\sigma}_n^m(t)\right)^n}$, where $n \sim 4$.

8.2 Suggestions for Future Research

This work makes important contributions to the analysis of time series of currency exchange rates through the examination of the behaviour of the spectral parameter, and reveals new areas for future research. In terms of interdisciplinary research, using the results obtained as a working tool, it is quite rational to turn first to a more detailed classification for crisis detection, particularly from the point of financial/economical estimation of their emergence and subsequent analysis of cause-and-effect relations. It will then be necessary both to use data for intermarket and between-market analysis, and apply additional methods of financial/economical analysis. Figure 8.1 generally support the possibility of such analyses.

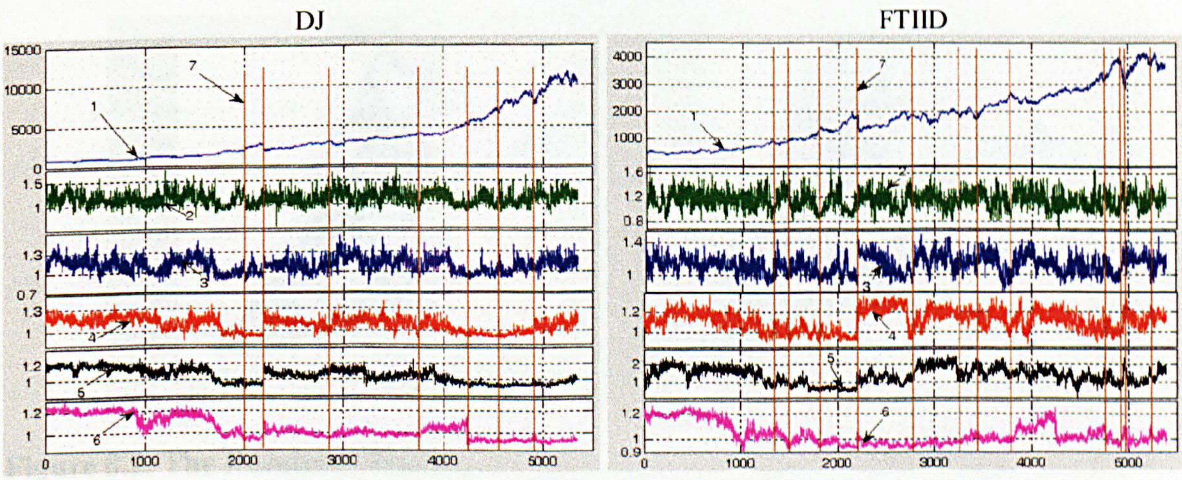


Figure 8.1. $q_q^m(t)$**, Shifted to the Beginning of the Reconstruction Window**

1: initial quotation; **2:** $m = 128$; **3:** $m = 256$; **4:** $m = 512$; **5:** $m = 1024$;

6: $m = 2048$; **7:** synchronised points of time

In terms of the within-disciplinary research, taking into account the possibility of identification of the critical areas of the phase space of the considered system, and the non-linear relations obtained between various components of the process, it will be useful to describe the behaviour of the system within the areas where values of the spectral parameter are small, using dynamic systems with small modes, and satisfying some projections with a small dimension. Within all the determining areas of the phase space of the considered system, it will be useful to more thoroughly analyse the established non-linear relations and to determine their inner properties via combining statistical methods of research with the methods of non-linear dynamics. On the boundaries of these areas it will be logical to consider various routes for moving into chaos. As the statistical relationship between some systematic and fluctuating components of the process is exponential, it will be useful to examine the possibility of applying models of fractional random processes (including multy-dimensional) in more detail, when analysing the fluctuating components (Figure 8.2).

$k = f\left(\frac{1}{\bar{\sigma}_n^m}\right)$ $\quad n = 128 \quad$ $k = f\left(\frac{1}{\left(\bar{\sigma}_n^m\right)^2}\right)$

$n = 256$

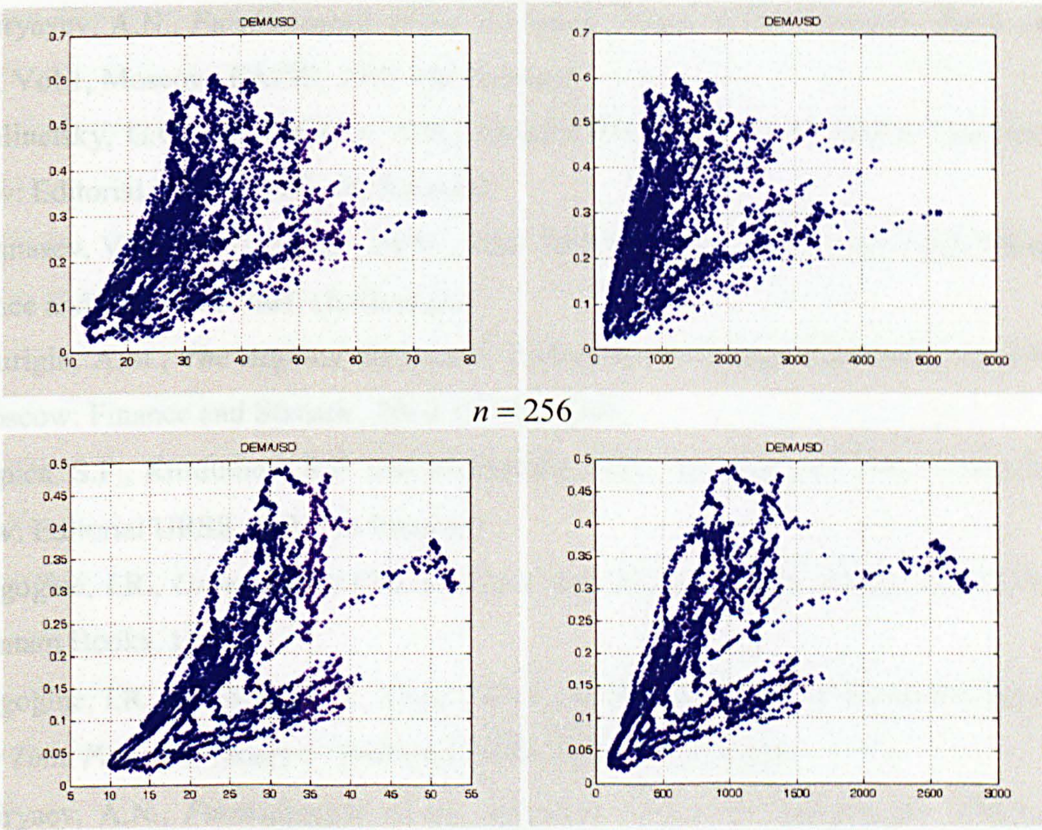


Figure 8.2. The Results of Finding $k = f\left(\frac{1}{\bar{\sigma}_n^m}\right)$, $k = f\left(\frac{1}{\left(\bar{\sigma}_n^m\right)^2}\right)$ for DEM/USD, $m = 1024$

With regards to the obtained results, it will be useful to consider models, based on non-Gaussian distributions. Future research has to be focused on the problem of forecasting. For this, it will be legitimate to obtain simulated time series of quotations for the corresponding values of the spectral parameter, e.g. via an application of the inverse Fourier transform, and to use these for the simulation of various types of Ito equations with subsequent transformations to the models of random processes with variable parameters.

References

1. Copeland, T.E., Weston, J.F. and Shastri, K., *Financial Theory and Corporate Policy*, 4th ed., Pearson Addison Wesley, 2003.
2. Martin, J.D., Cox Jr., S.H., McMinn, R.D. and Maminn R.D., *The Theory of Finance. Evidence and Applications*, International Thomson Publishing, 1997.
3. Menton, R.C., *Continuous-Time Finance*, Reprint edition, Blackwell Publishers, 1992.
4. Watsham, T.J. and Parramore, K., *Quantitative Methods in Finance*, Thomson Business Press, 1996.
5. Aivazian, S.A. and Mkhitarian, V.S., *Applied Statistics and Essentials of Econometrics*, Moscow: Book-Publishing Association UNITY, 1998.
6. Shiryayev, A.N., *Fundamentals of the Stochastic Financial Mathematics. Facts and Models*, Vol.1, Moscow: FAZIS, 1998. (In Russian)
7. Malinetsky, G.G. and Potapov, A.B., *Modern Problems of Non-Linear Dynamics*, Moscow: Editorial URSS, 2000. (In Russian)
8. Afanasev, V.N. and Uzbashev, M.M., *Time Series Analysis and Forecasting*, Moscow: Finance and Statistics, 2001. (In Russian)
9. Shurigin, A.M., *The Applied Stochastic: Robustness Property, Evaluation, and Forecast*, Moscow: Finance and Statistic, 2000. (In Russian)
10. Kapica, S.P., Kurdumov, S.P. and Malinetsky, G.G., *Synergy and Future Forecast*, Moscow: Editorial URSS, 2001 (In Russian)
11. Prigogine, I.R., *Order out of Chaos. Man's New Dialogue with Nature*, New York, N.Y.: Bantam Books, 1984.
12. Prigogine, I.R. and Stengers, I., *Time, Chaos and the Quantum – Towards the Resolution of the Time Paradox*, Moscow: Editorial URSS, 2003. (In Russian)
13. Shiryaev, A.N., *Fundamentals of the Stochastic Financial Mathematics. Theory*, Vol.2, Moscow: FAZIS, 1998. (In Russian)
14. Kamaev, V.A., *Mathematical Methods of Decision-Making in Economics*, Moscow: Finstatinform, 1999. (In Russian)
15. Pitev, U.P., *Probability. Theory and Application Elements*, Moscow: Editorial URSS, 2000 (In Russian)
16. Puu, T.O., *Non-Linear Economic Dynamics*, Springer-Verlag Berlin Heidelberg, 1997.
17. Zang, W.B., *Synergetic Economics: Time and Change in Non-Linear Economic. (Springer Series in Synergetics)*, Springer-Verlag Berlin Heidelberg, 1991.

18. Shelobaev, S.I., *Mathematical Methods and Models in Economics, Finance, Business*, Moscow: UNITI-DANA, 2000. (In Russian)
19. Berge, P., Pomo, I. and Vidal, K., *Order in Chaos. About Deterministic Approach in Turbulence*, Moscow: Mercury-Press, 2000. (In Russian, translation from French)
20. Kats, M., *Several Physical and Mathematical Probabilistic Problems*, Moscow: Mercury-Press, 2000 (In Russian)
21. Getling, A.V., *Rayleigh-Benard Convection: Structures and Dynamics*, World Scientific Pub Co, 1998.
22. Klimontovich, Y.L., *Turbulent Flow and Chaos Structure: A New Approach to the Static Theory of Open Systems*, Moscow: Nauka, Fizmatlit, 1990. (In Russian)
23. Malinetsky, G.G., *Chaos. Structures. Computing Experiment: Introduction to the Non-Linear Dynamics*, Moscow: Editorial, URSS, 2000 (In Russian)
24. Morozov, A.N., *Irreversible Processes and Brownian Motion: The Physical and Engineering Problems*, Moscow: Bauman Moscow State Technical University Press, 1997. (In Russian)
25. Morozov, A.N. and Nazolin, A.L., *Dynamic Systems with Fluctuating Time*, Moscow: Bauman Moscow State Technical University Press, 2001. (In Russian)
26. Ebeling, W., Engel, A. and Feistel, R., *Physics of Evolution Processes*, Moscow: Editorial, URSS, 2001. (In Russian)
27. Prigogine, I., *Non-Equilibrium Statistical Mechanics*, New York: John Wiley & Sons Inc., 1962.
28. Lichtenberg, A.J. and Lieberman, M.A., *Regular and Stochastic Motion: Applied Mathematical Sciences*, Springer-Verlag, 1983.
29. Murphy, J.J., *Intermarket Technical Analysis. Trading Strategies for the Global Stock, Bond, Commodity and Currency Market*, Wiley Finance Editions, John Wiley & Sons Inc., 1991.
30. Murphy, J.J., *Technical Analysis of the Futures Markets: A Comprehensive Guide to Trading Methods and Applications*, New York Institute of Finance, Prentice-Hall Press, 1999.
31. DeMark, T.R., *The New Science of Technical Analysis*, New York: John Wiley & Sons Inc., 1994.
32. Tikhonov, A.N. and Samarsky, A.A., *Equations of Mathematical Physics*, Moscow: Nauka, 1977. (In Russian)
33. Landau, L.D. and Lifshitz, E.M., *Course of Theoretical Physics*, Vol. 1-10, Moscow: Nauka, 1988. (In Russian)

34. Whittaker, E.T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th ed., Cambridge: University Press, 1988.

35. Pugachev, V.S., *Lectures on Functional Analysis*, Moscow: MAI, 1996. (In Russian)

36. Tabor, M., *Chaos and Integrability in Non-Linear Dynamics*, New York: Wiley-Interscience, 1989.

37. Crownover, R.M., *Introduction to Fractals and Chaos*, Jones & Bartlett Publishers, 1995.

38. Schroeder, M.R., *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*, W.H. Freeman & Company, 1992.

39. Paitgen, H.-O. and Richter, P.H., *The Beauty of Fractals: Images of Complex Dynamical Systems*, Springer-Verlag, 1986.

40. *Hydrodynamic Coordination of Particles in Suspension. Collection of Articles*, Moscow: Mir, 1980. (In Russian)

41. Protodyakonov, I.O. and Bogdanov, S.P., *The Statistical Theory of Transfer Phenomena in Applied Chemistry Processes*, Leningrad: Khimiya, 1983. (In Russian)

42. Wigner, E.P., *Symmetries and Reflections: Scientific Essays*, London: Greenwood Press, 1978.

43. Loycyansky, L.G., *Mechanics of Liquids and Gas*, Moscow: Nauka, 1987. (In Russian)

44. Klimontovich, F.L., *Statistical Physics*, Moscow: Nauka, 1982. (In Russian)

45. Dimentberg, M.F., *Random Processes in Dynamic Systems with Changing Parameters*, Moscow: Nauka, 1989. (In Russian)

46. Bukina, M.K. and Semenov, A.M., *Fundamentals of Exchange Relationships*, Moscow: Yurait, 1999. (In Russian)

47. Campbell, C.D., Campbell, R.G. and Dolan, E.G., *Money, Banking and Monetary Policy*, London: The Dryden Press, 1987.

48. Kravchenko, P.P., *How not to Lose on Financial Markets?*, Moscow: IAUC NAUFOR, 1999. (In Russian)

49. Zhvakolyuk, U.V., *Forex Intraday Trade*, SpB: Piter, 2000. (In Russian)

50. Irlikh, A.A., *Technical Analysis of Trade and Financial Markets*, Moscow: Infra-M, 1997. (In Russian)

51. Guillaume, D.M., Dacorogna, M.M., Davé, R.D., Müller, U.A., Olsen, R.B. and Pictet, O.V., From the Bird's Eye to the Microscope: A Survey of New Stylised Facts of the Intra-daily Foreign Exchange Markets, *Finance and Stochastic*, Vol.1(2), 1997, pp. 95-129.

52. *Proceedings of the First International Conference on High Frequency Data in Finance*, Introductory Lecture and 4 volumes. Zurich: Olsen & Associates, Research Institute for Applied Economics, 1995.

53. Dacorogna, M.M., Müller, U.A., Nagler, R.J., Pictet, O.V. and De Vries, C.G., The Distribution of Extremal Foreign Exchange Rate Returns in Extremely Large Data Sets, *Working Papers*, Zurich: Olsen & Associates, Research Institute for Applied Economics, 1995.

54. Dacorogna, M.M., Müller, U.A., Nagler, R.J., Olsen, R.B. and Pictet, O.V., A Geographical Model for the Daily and Volatility in Foreign Exchange Market, *Journal of International Money and Finance*, 12(4), 1993, pp. 413-438.

55. Müller, U.A., Dacorogna, M.M., Davé, R.D., Olsen, R.B., Pictet, O.V. and von Weizsacker, J.E., Volatilities of Different Time Resolutions – Analysing the Dynamics of Market Components, *Journal of Empirical Finance*, 4 (2–3), 1997, pp. 213–239.

56. Müller, U.A., Dacorogna, M.M., Davé, R.D., Olsen, R.B., Pictet, O.V. and Ward, J.R., Fractals and Intrinsic Time – A Challenge to Econometricians, *Working Papers*, Zurich: Olsen & Associates, Research Institute for Applied Economics, 1993.

57. Müller, U.A., Dacorogna, M.M., Olsen, R.B., Pictet, O.V., Schwarz, M. and Morgenegg, C., Statistical Study of Foreign Exchange Rates, Empirical Evidence of a Price Change Scaling Law, and Intra-day Analysis, *Journal of Banking and Finance*, 14, 1990, pp. 1189-1208.

58. Peters, E.E., *Chaos and Order in the Capital Markets: A New View of Cycles, Price, and Market Volatility*, Wiley Finance Editions, New York: John Wiley & Sons Inc., 1996.

59. Peters, E.E., *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*, New York: Wiley, 1994.

60. Ghashghaie S., Breymann W., Peinke J., Talkner P. and Dodge Y., Turbulent Cascades in Foreign Exchange Markets, *Nature*, 381, 1996, pp. 767-770.

61. Williams, B.M., *New Trading Dimensions: How to Profit from Chaos in Stocks, Bonds, and Commodities*, John Wiley & Sons Inc., 1998.

62. Colby, R.W. and Meyers, T.A., *The Encyclopedia of Technical Market Indicators*, McGraw-Hill Education, 2002.

63. Fisher, I., *The Theory of Interests: As Determined by Impatience to Spend Income and Opportunity of Invest It*, Porcupine Pr, 1930.

64. Miller, M.H. and Modigliani, F., The Cost of Capital, Corporation Finance, and the Theory of Investment, *American Economic Review*, 48(6), 1958, pp. 261-297.

65. Miller, M.H. and Modigliani, F., Dividend Policy, Growth, and the Valuation of Shares, *Journal of Business*, 34(10), 1961, pp. 411-433.
66. Markowitz, H.M., Portfolio selection, *Journal of Finance*, 7 (March), 1952, pp. 77-91.
67. Sharpe, W.F., Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risks, *Journal of Finance*, 19(9), 1964, pp. 425-442.
68. Ross, S.A., The Arbitrage Theory of Capital Asset Pricing, *Journal of Economic Theory*, 13, 1976, pp. 341-360.
69. Kendall, M.G., The Analysis of Economic Time-Series, Part 1, Prices, *Journal of the Royal Statistical Society*, 96, 1953, pp. 11-25.
70. Bachelier, L., Theorie de la Speculation, *Annales de l'Ecole Normale Superieure*, 17, 1900, pp. 21-86.
71. Korolyuk, V.S., Portenko, N.I., Skorohod, A.V. and Turbin, A.F., *A Guide to Probability Theory and Mathematical Statistics*, Moscow: Nauka, 1987. (In Russian)
72. Prokhorov, U.V. and Rozanov, U.A., *Probability Theory. Key Concepts. Limit Theorems. Random Processes*, Moscow: Nauka, 1987. (In Russian)
73. Black, F. and Scholes, M., The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81(3), 1973, pp. 637-659.
74. Merton, R.C., Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science*, 4(4), 1973, pp. 141-183.
75. Samuelson, P.A., Rational theory of Warrant Pricing, *Industrial Management Review*, 6, 1965, pp. 13-31.
76. Cowles, A., Can Stock Market Forecasters Forecast?, *Econometrica*, 1, 1933, pp. 309-324.
77. Cowles, A., Stock Market Forecasting, *Econometrica*, 12(3/4), 1944, pp. 206-214.
78. Working, H., A Random-Difference Series for Use in the Analysis of Time Series, *Journal of American Statistical Association*, 29, 1934, pp. 11-24.
79. Cowles, A. and Jones, H.E., Some a Posteriori Probabilities in Stock Market Action, *Econometrica*, 5, 1937, pp. 280-294.
80. Roberts, H.V., Stock-Market "Patterns" and Financial Analysis: Methodological Suggestions, *Journal of Finance*, 14, 1959, pp. 1-10.
81. Osborne, M.F.M., Brownian Motion in the Stock Market, *Operations Research*, 7, 1959, pp. 145-173.
82. Fama, E.F., The Behaviour of Stock Market Prices, *Journal of Business*, 34, 1965, pp. 420-429.

83. Mandelbrot, B.B., The Stable Paretian Income Distribution, when the Apparent Exponent is Near Two, *International Economic Review*, 4, 1963.

84. Rogers, L.R.C., Arbitrage with fractional Brownian motion, *Mathematical Finance*, 7, 1997, pp. 95-105.

85. Brockwell, P.J and Davis, R.A., *Time Series: Theory and Methods*, 2nd ed., New York: Springer-Verlag, 1991.

86. Hamilton, J.D., *Time Series Analysis*, Princeton, NJ: Princeton University Press, 1994.

87. Hannan, H.J., *Multiple Time Series*, New York: Wiley, 1970.

88. Mills, T.C., *The Econometric Modelling of Financial Time Series*, Cambridge: Cambridge University Press, 2nd ed., 1999.

89. Taylor, S., *Modelling Financial Time Series*. New York, Wiley, 1986.

90. Rabiner, L.R., *Theory and Application of Digital Signal Processing*, Prentice Hall, 1975.

91. Hardle, W., *Applied Nonparametric Regression (Econometric Society Monographs)*, Cambridge University Press, 1992.

92. Pesaran, M.H. and Potter, S.M., Eds., *Non-Linear Dynamics, Chaos, and Econometrics*, New York: John Wiley & Sons, 1993.

93. Tong, H., *Non-Linear Time Series: A Dynamical System Approach*, Oxford University Press, Reprint edition, 1993.

94. Rabinovich, M.I and Trubeckov, D.I., *Introduction to the Theory of Oscillations and Waves*, Moscow: Nauka, 1984. (In Russian)

95. Engel, R. F., Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of the United Kingdom Rate of Inflation, *Econometrica*, 50(4), 1982, pp. 987-1008.

96. Bollerslev, T., Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics*, 31, 1986.

97. Nelson, D.B., Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica*, 59, 1991, pp. 347-370.

98. Dacorogna, M.M., Muller, U.A., Embrechts, P. and Samorodnitsky, G., *Moment Conditions for the HARCH(k) Models*, Zurich: Olsen & Associates, , 1995.

99. Baillie, R.T., Bollerslev, T. and Mikkelsen, H.-O., Fractionally Integrated Generalised Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics*, 74, 1996, pp. 3-30.

100. Hurst, H., Long-term Storage Capacity of Reservoirs, *Transactions of American Society of Civil Engineers*, 116, 1951, pp. 770-808.

101. Feller, W., The Asymptotic Distribution of the Range of Sums of Independent Random Variables, *Annals of Mathematical Statistics*, 22(3), 1951, pp. 427-432.

102. Samorodnitsky, G. and Taqqu, M.S., *Stable Non-Gaussian Random Processes*, New York: Chapman&Hall, 1994.

103. Feller, W., *An Introduction to Probability Theory and Its Applications*, Wiley Text Books, 2 volumes, 1971.

104. Mandelbrot, B.B., *The Fractal Geometry of Nature*, San Francisco: W.H. Freeman, 1982.

105. Mandelbrot, B.B., *Fractals and Scaling in Finance*, Springer-Verlag, 1997.

106. Block, W.A., Hsieh, D.A. and LeBaron, B., *Non-Linear Dynamics, Chaos, and Instability: Statistical Theory and Economic Evidence*, Cambridge, MA: The MIT Press, 1991.

107. Devaney, R.L., *An Introduction to Chaotic Dynamical System*, Westview Press, 2nd ed., 1989.

108. Granger, C.W.J. and Terasvirta, T., *Modelling Non-Linear Economic Relationships*, Oxford: Oxford University Press, 1993.

109. Danilov, Y.A., *Lectures about nonlinear dynamics*, Moscow: Postmarket, 2001. (In Russian).

110. Vasicek, O., An Equilibrium Characterisation of the Term Structure, *Journal of Financial Economics*, 5, 1977, pp. 177-188.

111. Dothan, L., On the Term Structure of Interest Rates, *Journal of Financial Economics*, 6, 1978, pp. 59-69.

112. Cox, J.C., Ingersoll Jr., J.E. and Ross, S.A., A Theory of the Term Structure of Interest Rates, *Econometrica*, 53(2), 1985, pp. 385-408.

113. Ho, T.S.Y. and Lee, S.B., Term Structure Movements and Pricing Interest Rate Contingent Claims, *Journal of Finance*, 41, 1986, pp. 1011-1029.

114. Black, F.T., Derman, E. and Toy, W., A One Factor Model of Interest Rates and Its Application to Treasury Bond Options, *Financial Analysts Journal*, 1990, pp. 33-39.

115. Hull, J. and White, A., Pricing Interest Rate Derivative Securities, *The Review of Financial Studies*, 3(5), 1990, pp. 573-592

116. Black, F. and Karasinski, P., Bond and Option Pricing when Short Rates are Lognormal, *Financial Analysts Journal*, 47, 1991, pp. 52-59.

117. Sandmann, K. and Sondermann, D., A Term Structure Model and the Pricing of Interest Rate Derivatives, *The Review of Future Markets*, 12(2), 1993, pp. 391-423.

118. Chen, L., *A Three-Factor Model of the Term Structure of Interest Rates*, Washington, USA: Federal Reserve Board, 1995.

119. Mandelbrot, B.B. and Van Ness J.W., Fractional Brownian Motions, Fractional Noises and Applications, *SIAM Review*, 10(4), 1968, pp. 422-437.

120. Rees, C.S., Shah, S.M. and Stanojevic, C.V., *Theory and Applications of Fourier Analysis*, New York: Marcel Dekker, 1981.

121. Compte, A., Stochastic Foundation of Fractional Dynamics, *Physical Review E*, 53(4), 1996, pp. 4191-4193.

122. Blackledge, J.M., Turner, M., *Digital Signal Processing: Mathematical and Computational Methods, Software Development and Applications*, Chichester: Horwood Publishing, 2001.

123. Blackledge, J.M., On the Statistics of Dimentions: Fractal Modulation and Quantum Fractional Dynamics, *Microbar Security Limited*, 1999.

124. Tsallis, C., De Souza, A.M.C. and Maynard, R., *Derivation of Levy – Type Anomalous Superdiffusion from Generalised Statistical Mechanics*. In: Shlesinger, M.F., Zaslavsky, G. M. and Frisch, R., Eds., Levy Flights and Related Topics in Physics, Springer, 1944, pp. 269-289.

125. Mikhailov, S., *Fractal Modulation and Encryption*, PhD Thesis, De Montfort University, 1999.

126. Novickiy, P.V. and Zorgaf, I.A., *Errors Estimation of Measuring Results*, Leningrad: Energoatomizdat, 1991. (In Russian)

127. Lvovskiy, E.V., *Statistical Methods of Construction of Empirical Formulas*, Moscow: Visshaya Shkola, 1988. (In Russian)

128. Mandelbrot, B.B., *Fractals: Form, Chance, and Dimension*, San Francisco: W.H. Freeman, 1977.

129. Mandelbrot, B.B., Limit Theorems on the Self-normalized Range of Weakly and Strongly Dependent Processes, *Zeitschrift Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 31, 1975, pp. 271-285.

130. Mandelbrot, B.B. and Wallis, J.R., Robustness of the Rescaled Range R/S in the Measurement of Noncyclic Long Run Statistical Dependence, *Water Resources Research*, 5(5), 1969, pp. 967-988.

131. Mandelbrot, B.B., Some Noises with 1/f Spectrum: a Bridge between Direct Current and White Noise, *IEEE Transactions on Information Theory*, IT-13, 1967, pp. 289-298.

132. Mandelbrot, B.B., Statistical Methodology for Non-periodic Cycles: From the Covariance to R/S Analysis, *Annals of Economic and Social Measurement*, 1(3), 1972, pp. 259-290.

133. Mandelbrot, B.B., The Pareto-Levy Law and the Distribution of Income, *International Economic Review*, 1, 1960, pp. 79-106.

134. Mandelbrot, B.B., The Variation of Certain Speculative Prices, *Journal of Business*, 36, 1963, pp. 394-416.

135. Mandelbrot, B.B., Une Classe de Processus Stochastiques Homothétiques à Soi; Application á la Loi Qlimatologique de H.E. Hurst, *Comptes Rendus des Séances de l'Académie des Sciences de Paris*, 240, 1965, pp. 3274-3277.

136. Mandelbrot, B.B. and Taylor, H.M., On the Distribution of Stock Price Difference, *Operations Research*, 15(6), 1967, pp. 1057-1062.

137. Mandelbrot, B.B. When Can Price be Arbitraged Efficiently? A Limit of the Validity of the Random Walk and Martingale Models. *Review of Economics and Statistics*, 53, 1971, pp. 225-236.

138. Mandelbrot B.B. and Wallis, J.R., Computer Experiments with Fractional Gaussian Noises, *Water Resources Research*, 5, 1969, pp. 228-267.

139. Packard, N.H., Crutchfield, J. P., Farmer, J. D. and Shaw, R. S., Geometry from a Time Series, *Physics Review Letters*, 45, 1980, pp. 712-716.

140. Takens, F., Detecting Strange Attractors in Turbulence, *Lecture Notes in Math.*, Berlin: Springer-Verlag, 898, 1980, pp. 336-381.

141. Sternberg, S., *Lectures on Differental Geometry*, Chelsea Pub Co, 2nd ed., 1999.

142. Temam, R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, 2nd ed., 1997.

143. Broomhead, D.S. and Jones, R., Time-Series Analysis, *Proceedings of the Royal Society of London*, 423, 1989, pp.103-121.

144. Abarbanel, H.D. and Kennel, M.B., Local False Nearest Neighbours and Dynamical Dimensions from Observed Chaotic Data, *Physical Review E*, 47, 1993, pp. 3057-3068.

145. Haken, H., *Synergetics: Introduction and Advanced Topics*, Springer-Verlag, 2004.

146. Hediger, T., Passamante, A. and Farrell, M.E., Characterizing Attractors Using Local Intrinsic Dimensions Calculated by Singular-Value Decomposition and Information-Theoretic Criteria, *Physical Review A*, 41, 1990, pp. 5325-5332.

147. Bayly, B.J., Goldhirsch, I. and Orszag, S.A., Independent Degrees of Freedom of Dynamical Systems, *Journal of Scientific Computing*, 2, 1987, pp. 111-121.

148. Leibert, W., Pawelzik, K. and Schuster, H.G., Optimal Embedding of Chaotic Attractors from Topological Considerations, *Europhysics Letters*, 14, 1991, pp. 521-526.

149. Fraser, A.M., Reconstructing Attractors from Scalar Time Series: A Comparison of Singular System and Redundancy Criteria, *Physica D*, 34, 1989, pp. 391-404.

150. Gibson, J.F, Farmer, J.D., Casdagli, M. and Eubank, S., An Analytic Approach to Practical State Space Reconstruction, *Physica D*, 57, 1, 1992.

151. Fraser, A.M., Information and Entropy in Strange Attractors, *IEEE Transactions on Information Theory*, 35, 1989, pp. 245-262.

152. Fraser, A.M. and Swinney, H.L., Independent Coordinates for Strange Attractors from Mutual Information, *Physical Review A*, 33, 1986, pp. 1134-1140.

153. Box, G., Jenkins, G.M. and Reinsel, G., *Time Series Analysis: Forecasting and Control*, Prentice Hall, 3rd ed., 1994.

154. Ivahnenko, A.G., *Random Processes Forecasting*, Kiev: Naukova Dumka, 1971. (In Russian)

155. Grassberg, P. and Procaccia, I., Estimation of the Kolmogorov Entropy from a Chaotic Signal, *Physical Review A*, 28, 4, 1983, pp. 2591-2593.

156. Grassberg, P., An Optimized Box-assisted algorithm for Fractal Dimensions, *Physics Letters A*, 148, 1990, pp. 63-68.

157. Theiler, J., Estimating Fractal Dimension, *Journal of the Optical Society of America A*, 7, 1990, pp. 1055-1073.

158. Malinetsky, G.G. and Potapov, A.B., About Calculation of Dimension of Strange Attractors. *Journal of the Calculus Mathematics and Mathematical Physics*, 28, 7, 1988, pp. 1021-1037. (In Russian)

159. Smith, R.L., Estimating Dimension in Noisy Chaotic Time Series, *Journal of the Royal Statistical Society B*, 54(2), 1992, pp. 329-351.

160. Balakrishnan, V., Nicolis, C. and Nicolis, G., Extreme Value Distributions in Chaotic Dynamics, *Journal Statist. Phys.*, 80, 1995, pp. 307-336.

161. Corn, G. and Corn, T., *Mathematical directory*, Moscow: Nauka, 1977. (In Russian)

162. Theiler, J., Some Comments on the Correlation Dimension of $1/f^{\alpha}$ Noise, *Physics Letters A*, 155, 1991, pp. 480-493.

163. The data were purchased by De Montfort University from Global Financial data (http://www.globalfindata.com) in 1999.